A Note on Hausdorff Geometry of Polynomials

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1 Introduction

Geometry of polynomials study, among others, how the locations of the critical points of a polynomial, in the complex plane $\mathcal{C}$, depend on the locations of the zeros of this polynomial, see M. Marden [8]. Typical in this field are the Roll's type theorems in $\mathcal{C}$ [10], as

Theorem 1 (Grace–Heawood Theorem). *Let p be a polynomial of degree n and $p(z_1) = p(z_2) = 0$. Then there exists a critical point ζ_1 of p ($p'(\zeta_1) = 0$), which lies in the disk $D(c, r)$ with center $c = (z_1 + z_2)/2$ and radius $r = (|z_1 - z_2|/2)\cot(\pi/n)$.*

Grace–Heawood Theorem locates one critical point, using the positions of two zeros. There are other types of theorems, which locate the set of all critical points $A'(p)$ of a polynomial p, using the position of the set $A(p)$ of all zeros of this polynomial. Such theorems are, for example, the Gauss–Lucas Theorem and the Walsh's three circles theorem, see [10, Th. 3.2]. Formulating statements involving sets of points in $\mathcal{C}$, we may use the Hausdorff distance (see [6], [17]). It is natural to call *Hausdorff geometry of polynomials* this part of the Geometry of polynomials, which studies the location of the set of the critical points, using the set of all zeros of a polynomial.

1.1 Hausdorff distance in the complex plane $\mathcal{C}$

Let A and B be closed and bounded sets in $\mathcal{C}$. Set $\rho(z; A) = \inf\{|z - \zeta| : \zeta \in A\}$ and call $\rho(A; B) = \sup\{\rho(z; B) : \zeta \in A\}$ *Hausdorff deviation* of A from B. Then

$$h(A, B) = \max\{\rho(A; B), \rho(B; A)\}$$

is called *Hausdorff distance* between A and B (see [6], [17]).

The Hausdorff distance is a metric in the space of all closed and bounded subsets of $\mathcal{C}$.

For convenience, we introduce some notions connected with a set on the plane. The smallest convex point set $H(A) \subset \mathcal{C}$ containing the set A is called the *convex hull* of A. Let H be a closed and convex set in $\mathcal{C}$. A line l is called *support line* of H, if l and H have at least one common point and H lies only in one of the closed half-planes defined by l. Let $0 \leq \varphi < \pi$. Denote by $l_1(H, \varphi)$ the support line of H, which is over-right of H and the elementary angle from the positive real axis to l_1 is equal to φ; and by $l_2(H, \varphi)$ the support line of H, which is below-left of H and the elementary angle from l_2 to the negative real axis is equal to φ. Call the line $l_0(H, \varphi)$, which lies in the mid between $l_1(H, \varphi)$ and $l_2(H, \varphi)$, a *central line* of H.

Every central line $l_0 = l_0(H, \varphi)$ of H divides this set in two closed *halfs* $H_1(\varphi)$, between l_1 and l_0, and $H_2(\varphi)$, between l_0 and l_2.

Let A and B be bounded and closed point sets in the complex plane. Then

$$\rho(H(A); H(B)) = \rho(\partial H(A); \partial H(B)) \leq \rho(A; H(B)) \leq \rho(A; B), \tag{1.1}$$

where ∂A is the boundary of A.

1.2 Polynomial pairs

For every polynomial $p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$ and $p'(z) = n(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_{n-1})$, denote by:

1. $A(p) = \{z_1, z_2, \ldots, z_n\}$ - the set of all zeros of p, some of them may coincide, if p has multiple zeros.
2. $A'(p) = \{\zeta_1, \zeta_2, \cdots, \zeta_{n-1}\}$ - the set of all critical points of p.
3. $D(p) = D(c(p), r(p))$ - the *circum disk* of $A(p)$, (the smallest disk, which contains $A(p)$), where $c(p)$ is the *circum center* and $r(p)$ is the *circum radius* of $A(p)$.
4. $C(p) = C(c(p), r(p))$ - the circum circle of $A(p)$.
5. $\hat{A}(p) = C(p) \bigcup H(A(p))$ - the set of all *border* zeros of p.
6. $\mathcal{P}$ - the set of all monic polynomials p of degree $n \geq 3$ with $c(p) = 0$ and $r(p) = 1$.
7. $\mathcal{P}_n$ - the set of all monic polynomials $p \in \mathcal{P}$ of degree n.

A pair of sets $\{A, A'\}$ is called a *polynomial pair* if $A = A(p)$ and $A' = A'(p)$ for a polynomial $p \in \mathcal{P}$.

From the definition of $\mathcal{P}$ there follows a simple and useful statement.

Lemma 1 *If $p \in \mathcal{P}$, then:*

(i) *At least two zeros of p lie in C.*

(ii) *At least one zero of p lies in each subarc of C with length π, where C is the unit circle.*

2 Hausdorff polynomial geometry

A classical theorem from the Hausdorff polynomial geometry is

Theorem 2 (Gauss-Lucas). *For every polynomial p, the equality* $\rho(A'(p); H(A(p))) = 0$ *holds.*

More than 40 years ago, we formulated

Conjecture 1 *For every polynomial p, the inequality $\rho(A(p); A'(p)) \leq r(p)$ holds.*

This conjecture was published in the Hayman's book [7, problem 4.5.], by mistake, as *Ilieff's Conjecture* (see [9, p. 267]). Until the end of XXth century, Conjecture 1 was not proved or disproved, after more than 70 concerning papers were published. In 1999, Conjecture 1 was proved by Brown, Johnny E. and Xiang, Guangping for polynomials of degree $n \leq 8$ [5]. The cases $n = 3, 4, 5, 6, 7$ were proved one after the other, see for example [3], [14], [11], [2] and [18]. Many partial proofs of Conjecture 1, for special types of polynomials or for special location of the considered zero, were published. It is impossible to list all them here.

In 1985, Abdul Aziz [1] proved

Theorem 3 (**Aziz' Theorem**). *For every polynomial p, the inequality* $\rho(A'(p); A(p)) \leq r(p)$ *holds.*

Having in mind Aziz's Theorem, we may formulate Conjecture 1 in the form: *For every polynomial p, the inequality $h(A(p); A'(p)) \leq r(p)$ holds.*

In this paper we consider, besides the set of all zeros $A(p)$ and the set of all critical points $A'(p)$, also their convex hulls and their convex hull circum disks. In the following we consider only polynomials from $\mathcal{P}$ and $\mathcal{P}_n$, which is sufficient as the problems in Hausdorff polynomial geometry are invariant with respect to rotation, translation and scaling.

We may expect that in a polynomial pair $\{A, A'\}$ the most removed from A' are the border zeros. This may be true, but the fact is that for all border zeros, Conjecture 1 is true [14]. Also, we have

Theorem 4 (G. Schmeisser (1969) [15, Satz 1.]). *Conjecture 1 is true for $p \in \mathcal{P}$ if* $\hat{A}(p) = A(p)$.

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2.1 Hausdorff deviations between the sets $A(p)$, $A'(p)$, $H(A(p))$ and $H(A'(p))$

For a polynomial pair $\{A(p), A'(p)\} = \{A, A'\}$ with $p \in \mathcal{P}$, we shall consider estimations of all Hausdorff deviations between the four sets $A, A', H(A)$ and $H(A')$. There are all together 12 such deviations. Four of them are trivial:

$\rho(A; H(A)) = \rho(A'; H(A')) = 0, \quad \rho(H(A); A) \leq 1$ and $\rho(H(A'); A') \leq 1$.

The estimate $\rho(A'; A) \leq 1$ is Aziz's Theorem.

The estimate $\rho(H(A'); A) \leq 1$ follows from Aziz's Theorem.

The estimate $\rho(A'; H(A)) = 0$ is Gauss-Lucas Theorem.

The estimate $\rho(A; A') \leq 1$ is Conjecture 1.

It is easy to see that $\rho(A; H(A')) = \rho(H(A); H(A'))$. As $\rho(A; H(A')) \leq \rho(A; A')$, see (1.1), it is safe to formulate

Conjecture 2 *For every polynomial pair $\{A, A'\}$, the inequality $\rho(A; H(A')) \leq 1$ holds.*

Conjecture 2 is weaker than Conjecture 1, but it may be not easier.

For the estimate $\rho(H(A); A')$, in 1977, G. Schmeisser [16] formulate

Conjecture 3 (Schweisser's Conjecture). *For every polynomial pair $\{A, A'\}$, the inequality $\rho(H(A); A') \leq 1$ holds.*

Conjecture 3 is stronger than Conjecture 1. From the proof of Theorem 4 follows that Conjecture 3 is true for a polynomial pair $\{A, A'\}$ if $\hat{A} = A$.

2.2 Hausdorff deviations involving the circum disk

Problem 1 *For $p \in \mathcal{P}_n$, estimate the deviations $\rho(D, A'(p))$ and $\rho(D, H(A'(p)))$.*

It is easy to figure out that
$\rho(D, A(p)) \leq \sqrt{2}$ and $\rho(D, H(A(p))) \leq 1$ for every polynomial $p \in \mathcal{P}$, see Lemma 1.

Let $\rho'_n = \sup\{\rho(D, A'(p)) : \ p \in \mathcal{P}_n\}$ and $\rho''_n = \sup\{\rho(D, H(A'(p))) : \ p \in \mathcal{P}_n\}$. It is obvious that $\rho'_2 = \rho''_2 = 1$, see Lemma 1, and $\rho''_n \leq \rho'_n$, see (1.1).

For the polynomial $p(z) = (z^2+1)(z+1)^{n-2}$ we calculate that

$$\rho(D; A'(p)) = \sqrt{2} \quad \text{for} \quad n = 3, 4, \dots \ ,$$

therefore $\rho'_n \geq \sqrt{2}$ for every $n \geq 3$.

Conjecture 4 *For every $p \in \mathcal{P}$ of degree ≥ 3, the inequality $\rho(D; A'(p)) \leq \sqrt{2}$ holds and $\rho' = \sup\{\rho'_n : \ n \geq 3\} = \sqrt{2}$.*

It is natural to expect that if a polynomial $p \in \mathcal{P}$ is extremal for the deviations $\rho(D; A'(p))$ and $\rho(D; H(A'(p)))$, then $A(p)$ lies in one closed half of D.

Problem 2 *For $p \in \mathcal{P}_n$, estimate the deviations $\rho(D, D(p'))$.*

It is trivial that $\rho(D, D(p)) = 0$, as by definition, if $p \in \mathcal{P}_n$, then $D(p) = D$. From the Gauss–Lucas Theorem there follows that $\rho(p', D(p)) = 0$. Having in mind the polynomial $p(z) = z^n - 1$, we formulate

Conjecture 5 *Fore every $p \in \mathcal{P}$, the inequality $\rho(D(p'), D(p) \leq 1$ holds.*

Evidently, Conjecture 5 is weaker than Conjecture 2, but still not easy.

2.3 Extremal polynomials

A polynomial is *extremal* for an inequality, if for this polynomial the inequality becomes equality. The existence of extremal polynomials in Hausdorff polynomial geometry is usually simple problem as $\mathcal{P}$ is a normal family.

The polynomial $p(z) = z^n - z_1$ with $|z_1| = 1$ is extremal in many estimations. It is extremal for Aziz's Theorem and if there are true, it is extremal for Conjecture 1, Conjecture 2, Conjecture 3 and Conjecture 5.

In 1972 Phelps, D. and R. S. Rodriguez [13] formulate

Conjecture 6 (Phelps–Rodriguez Conjecture). *If $p \in \mathcal{P}_n$ is extremal in $\mathcal{P}_n$, then $p(z) = z^n - |z_1|$ with $|z_1| = 1$.*

It is interesting to observe that the pair $\{A(p); A'(p)\}$ for $p(z) = z^n - 1$ has the following properties:

1. The measure $\mu(H(A(p))$ is as big as possible.
2. The measure $\mu(H(A'(p)))$ is as small as possible.
3. $\hat{A}(p) = A(p)$.

It is natural to expect that every other extramal pair will have similar properties. Interesting and deep investigations in this direction are made by M. J. Miller (1990) [12].

To study Problem 1, it is sufficient to consider only polynomials $p \in \mathcal{P}$ for which the support line $l_1' = l_1(H(A'(p)), \pi/2)$ cuts a maximal sector from the unit disk D. Let $\breve{\mathcal{P}} \subset \mathcal{P}$ be the set of all these polynomials.

It is easy to apprehend that if $p \in \breve{\mathcal{P}}$ and ξ is the crossing point of l_1' and the real axis, then there are two possibilities:

1) $\xi = \zeta_1 \in A'(p)$.

2) There are two points $\zeta_1, \zeta_2 \in A'(p) \bigcap l_1'$, such that $\Im(\zeta_1)\Im(\zeta_2) \leq 0$.

Let $l_1 = l_1(H(A(p)), \pi/2)$ and u be the crossing point of l_1 with the real axis. Then $u \geq 0$, see Lemma 1.

We finish with

Conjecture 7 *If p is an extremal polynomial for Problem 1, then $u = 0$.*

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