# On the Hausdorff Geometry of Polynomials, a Generalized Conjecture

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## 1 Introduction

In 1958, I formulated the following conjecture [10, 20].

**Conjecture 1** *If all the zeros of the polynomial* $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ $(n \geq 2)$ *lie in the unit disk* $D = D(0,1) = \{z : \ |z| \leq 1\}$*, then for every* $z_k$ *the disk* $D(z_k, 1) = \{z : \ |z - z_k| \leq 1\}$ *contains at least one zero of* $p'(z)$*.*

Up to now, this conjecture was neither proved nor disproved, although more than 80 related papers have been published on this topic. In attempts to prove this conjecture, many new results have been obtained and many other problems and conjectures have been formulated [20]. Our purpose in this paper is to formulate Conjecture 1 in terms of the Hausdorff deviation between two discreet sets of points. This equivalent formulation gives way to natural generalizations.

## 2 Preliminaries and notations

Let $A$ and $B$ be two bounded and closed sets of points in the complex plane $\mathcal{C}$. We will use the notations:

1. $\rho(b; A) = \inf\{|b - a| : \ a \in A\}$,

2. $\rho(B; A) = \sup\{\rho(b; A) : \ b \in B\}$ - the **Hausdorff deviation** of $B$ from $A$,

3. $h(A, B) = \max\{\rho(A; B), \rho(B; A)\}$ - the **Hausdorff distance** between $A$ and $B$.

4. $H(A)$ - the smallest closed and convex point set, which contains $A$, called the **convex hull** of $A$.

5. $D_c(A) = D(c_c(A), r_c(A))$ - the **circumself disk** of $A$, i. e., the smallest closed disk, which contains $A$, with $c_c(A)$ - the **circumself center** of $A$ and $r_c(A)$ - the **circumself radius** of $A$;

6. $C_c(A) = \{z : |z - c_c(A)| = r_c(A)\}$ - the **circumself circle**[1] of $A$;

The following relations are obvious for every $A \subset \mathcal{C}$:

$$A \subset H(A) \subset D_c(A). \tag{2.1}$$

A simple and useful statement is

**Lemma 2.1** *If $A, B, V$ are bounded point sets in $\mathcal{C}$ and $B \subset V$, then*

$$\rho(A; B) \geq \rho(A; V) \quad \text{and} \quad \rho(B; A) \leq \rho(V; A).$$

**Proof**. From the definition

$$\rho(A; B) = \sup\{\rho(a; B) : \ a \in A\} \geq \sup\{\rho(a; V) : \ a \in A\} = \rho(A; V),$$

as $B \subset V$ and

$$\rho(a; B) = \inf\{|a - b| : \ b \in B\} \geq \inf\{|a - v| : \ v \in V\} = \rho(a; V).$$

In the same way

$$\rho(V; A) = \sup\{\rho(v; A) : \ v \in V\} \geq \sup\{\rho(b; A) : \ b \in B\} = \rho(B; A),$$

as $B \subset V$. $\square$

In general $\rho(B; A) \neq \rho(A; B)$.

**Corollary 2.1** *If $A$ and $B$ are two bounded and closed point sets in the complex plane. Then $h(H(A), H(B)) \leq h(A, B)$.*

From the definition of circumself disk, there follows

**Lemma 2.2** *If $A \subset \mathcal{C}$ is a closed and bounded (and has at least two distinct points), then:*

1. *At least two distinct points of $A$ lie in $C_c(A)$;*
2. *At least one point of $A$ lies in each subarc of $C_c(A)$ with length $\pi$.*

[1] Note the difference between circumcircle of a triangle and circumself circle

## 2.1 Polynomial sets

As we are interested only in the relative location of the set of zeros af a polynomial and the set of zeros of its $s$-th derivative, it is sufficient to consider only monic polynomials $p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$. The geometric interrelation between the set $A(p)$ of the zeros of $p$ and the set $A(p^{(s)})$ of the zeros of $p^{(s)}$ is invariant with respect to rotation, translation and scaling. Studying this interrelation, we may prescribe the circumself center and the circumself radius of the set $A(p)$.

**Definition 2.1** *Set the following notations:*

1. $A(p)$ - *the set of all distinct zeros of the polynomial* $p$*;*
2. $H(p) = H(A(p))$ - *the* **convex hull** *of* $p$*;*
3. $D_c(p) = D_c(A(p)) = D(c_c(p), r_c(p))$ - *the* **circumself disk** *of* $p$ *with* $c_c(p) = c_c(A(p))$ - *the* **circumself center** *of* $p$ *and* $r_c(p) = r_c(A(p))$ - *the* **circumself radius** *of* $p$*;*
4. $\mathcal{P}_n$ - *the set of all monic polynomials* $p$ *of degree* $n \geq 2$*, with circumself disk* $D_c(p) = D(c_c(p), r_c(p)) = D(0, 1) = D$*;*

From Lemma 2.2 there follows:

**Corollary 2.2** *If* $p \in \mathcal{P}_n$*, then:*

1. *At least two zeros of* $p$ *lie in* $C$*;*
2. *At least one zero of* $p$ *lies in each subarc of* $C$ *with length* $\pi$*, therefore* $m \geq 2$*.*

## 2.2 Basic statements

First we list two classical theorems from the Geometry of polynomials. For more results see [13].

**Theorem 2.1** (Gauss–Lucas). *For every algebraic polynomial* $p$*,* $A(p') \subset H(p)$*.*

**Definition 2.2** *The polynomials*

$$p(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

*are called* **apolar** *if*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{2.2}$$

*The equation* (2.2) *is called the* **apolarity condition**.

**Theorem 2.2** (Grace Apolarity Theorem). *Let $p$ and $q$ be apolar, then any circular region containing all zeros of $p$ or $q$ contains at least a zero of the other.*

A circular region is a closed disk or a closed half-plain.

**Lemma 2.3** *If $p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$ and $a_0 \neq 0$, then $p$ has at least one zero in the disk $D(0, r)$, where $r = |a_m/a_0|^{1/m}$.*

**Proof**. The polynomial $q(z) = a_0 z^m - (-1)^m a_m$ is apolar to the polynomial $p$ and all of the zeros of $q$ lie in the disk $D(0, |a_m/a_0|^{1/m})$. The proof of the lemma follows from the Grace Apolarity Theorem 2.2. □

**Lemma 2.4** *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|p^{(s)}(z_1)| \leq (n!/(n-s)!) r^{n-s}$, $s = 1, 2, \ldots, n-1$, then there exists a $\zeta_1 \in A(p^{(s)})$, such that $|z_1 - \zeta_1| \leq r$.*

**Proof**. We have

$$p^{(s)}(z_1 - z) = (-1)^{n-s} \frac{n!}{(n-s)!} z^{n-s} + \cdots + p^{(s)}(z_1).$$

By Lemma 2.3

$$\left| \frac{a_{n-s}}{a_0} \right| = \left| \frac{p^{(s)}(z_1)(n-s)!}{n!} \right| = r^{n-s},$$

or $r = |a_m/a_0|^{1/(n-s)}$. □

# 3 Generalized conjecture

We may formulate Conjecture 1 in the following form.

- *For every $p \in \mathcal{P}_n$, the inequality $\rho(A(p); A(p')) \leq 1$ holds.*

A generalization of this conjecture is the following.

**Conjecture 2** *For every $p \in \mathcal{P}_n$ and $s \in \{1, 2, \ldots, n-1\}$, the inequality*

$$\rho(A(p); A(p^{(s)})) \leq \frac{2s}{s+1}$$

*holds.*

In this section we consider partial proofs of Conjecture 2.

For $n = 1$, Conjecture 2 is Conjecture 1. For $n = n-1$, Conjecture 2 is trivial, as $p^{(n-1)}(z)$ has only one zero $\zeta_1 = (z_1 + z_2 + \cdots + z_n)/n$. Then

$$|z_1 - \zeta_1| \leq \sup \left\{ z_1 - \frac{z_1 + z_2 + \cdots + z_n}{n} : \ |z_k| \leq 1, k = 1, 2, \ldots, n \right\} = \frac{2(n-1)}{n}.$$

**Lemma 3.1** *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|z_1| = 1$, then there exists a zero $\zeta_1$ of $p^{(s)}(z)$ on the disk $D(z_1/(s+1), s/(s+1)) \subset D(z_1, 2s/(s+1))$.*

**Proof**. Using the method of A. MEIR AND A. SHARMA [14], we may set $z_1 = 1$ and $p(z) = (z-1)q(z)$. Let $\zeta_1, \zeta_2, \ldots, \zeta_{n-s}$ be the zeros of $p^{(s)}$ and $\eta_1, \eta_2, \ldots, \eta_{n-s}$ be the zeros of $q^{(s-1)}$. By Gauss–Lucas Theorem 2.1 $|\eta_k| \le 1$; $k = 1, 2, \ldots, n-s$. If $p^{(s)}(1) = 0$, we are done. Let $p^{(s)}(1) \ne 0$, then

$$\frac{p^{(s+1)}(1)}{p^{(s)}(1)} = \frac{s+1}{s} \cdot \frac{q^{(s)}(1)}{q^{(s-1)}(1)}$$

and

$$\sum_{j=1}^{n-s} \Re\left(\frac{1}{1-\zeta_j}\right) = \frac{s+1}{s} \sum_{k=1}^{n-s} \Re\left(\frac{1}{1-\eta_k}\right) \ge \frac{(s+1)(n-s)}{2s}.$$

If $\Re(1/(1-\zeta_1)) = \max\{\Re(1/(1-\zeta_j)) :\ j = 1, 2, \ldots, n-s\}$, then

$$\Re\left(\frac{1}{1-\zeta_1}\right) \ge \frac{s+1}{2s},$$

therefore

$$\zeta_1 \in D(1/(s+1), s/(s+1)) \subset D(1, 2s/(s+1)). \qquad \square$$

**Corollary 3.1** *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|z_1| = 1$, then there exists $\zeta_1 \in A(p^{(s)})$, such that*

$$|z_1 - \zeta_1| \le \frac{2s}{s+1}; \quad s = 1, 2, \ldots, n-1.$$

**Corollary 3.2** *Conjecture 2 is true for a polynomial $p \in \mathcal{P}_n$ if all the corners of $H(p)$ are on the unit circle $C = \{z :\ |z| = 1\}$.*

**Proof**. If all the corners $z_{k_1}, z_{k_2}, \ldots, z_{k_l}$ of $H(p)$ are on the unit circle, then

$$H(p) \subset \bigcup_{m=1}^{l} D(z_{k_m}/(s+1), s/(s+1)).$$

If a zero $z_1 \in D(z_{k_1}/(s+1), s/(s+1)) \subset D(z_1, 2s/(s+1))$, then there exists a zero $\zeta_{k_1}$ of $p^{(s)}$, such that $|z_1 - \zeta_{k_1}| \le 2s/(s+1)$. $\square$

**Lemma 3.2** *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$, $|z_1| < 1$ and $s \in \{1, 2, \ldots, n-1\}$, then*

$$|p^{(s)}(z_1)| \le \frac{(n-3)!}{(n-s)!} 2^{n-s-1} s(n^2 + s^2 - 3n - 3s + 4).$$

**Proof**. As $|z_1| < 1$, by Lemma 2.2, there are two zeros $z_2, z_3 \in A(p)$ such that $|z_1| = |z_2| = 1$,

$$\arg z_2 \leq \arg z_1 \leq \arg z_3 \quad \text{and} \quad 0 \leq \arg z_3 - \arg z_2 \leq \pi.$$

Then, it is geometrically obvious that

$$|z_1 - z_2||z_1 - z_3| \leq 2 \quad \text{and} \quad \left| z_1 - \frac{z_2 + z_3}{2} \right| \leq 1. \tag{3.3}$$

Using (3.3), we have

$$p^{(s)}(z_1) = s! \sum_{2 \leq k_1 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}}) =$$

$$s! \left[ (z_1 - z_2)(z_1 - z_3) \sum_{4 \leq k_3 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_3})(z_1 - z_{k_4}) \cdots (z_1 - z_{k_{n-s}}) + \right.$$

$$2\left(z_1 - \frac{z_2 + z_3}{2}\right) \sum_{4 \leq k_2 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_2})(z_1 - z_{k_3}) \cdots (z_1 - z_{k_{n-s}}) +$$

$$\left. \sum_{4 \leq k_1 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}}) \right]$$

and

$$|p^{(s)}(z_1)| \leq s! \left[ 2 \cdot 2^{n-s-2} \binom{n-3}{n-s-2} + 2 \cdot 2^{n-s-1} \binom{n-3}{n-s-1} + 2^{n-s} \binom{n-3}{n-s} \right] =$$

$$\frac{(n-3)!}{(n-s)!} 2^{n-s-1} s(n^2 + s^2 - 3n - 3s + 4). \qquad \square$$

**Theorem 3.1** *Conjecture 2 is true for:*

*1)* $s = 1$ *and* $n = 3, 4,$*;*

*2)* $s = 2$ *and* $n = 3, 4, 5, 6$*;*

*3)* $s = 3$ *and* $n = 4, 5, 6, 7, 8, 9$*;*

*4)* $n \geq 5$ *and* $2s \geq n$*.*

**Proof** From Corollary 3.1 and Lemma 3.2 follows that Conjecture 2 is true for a given pair $(s, n)$ if

$$s \frac{n^2 + s^2 - 3n - 3s + 4}{n(n-1)(n-2)} \leq 2 \left( \frac{s}{s+1} \right)^{n-s}. \tag{3.4}$$

For $s = 1$, the inequality (3.4) is $n \geq 2^{n-2}$, and that is an easy proof of Conjecture 1 fore $n = 3, 4$. Conjecture 1 is proved for $n \leq 8$, see [7].

For $s = 2$, the inequality (3.4) is $n \geq (3/2)^{n-2}$, which is satisfied for $n = 4, 5, 6, 7, 8, 9$.

For $2s = n$, the inequality (3.4) is

$$5 - \frac{3}{n-1} \leq 16\left(1 + \frac{2}{n}\right)^{-\frac{n}{2}},$$

or

$$5 \leq 16e^{-1} = 5.886\ldots. \qquad \square$$

# 4 Hausdorff deviations

For a polynomial $p \in \mathcal{P}_n$, we shall consider the deviation of one of the sets

$$A(p), H(p), D_c(p) = D \tag{4.5}$$

from one of the sets

$$A(p^{(s)}), H(p^{(s)}), D_c(p^{(s)}), \tag{4.6}$$

for $s = 1, 2, \ldots n-1$.

By definition

$$A(p) \subset H(p) \subset D_c(p) = D \quad \text{and} \quad .A(p^{(s)}) \subset H(p^{(s)}) \subset D_c(p^{(s)}). \tag{4.7}$$

If $\mathcal{U}$ is a compact set of polynomials, then the sets

$$A(\mathcal{U}) = \{A(p) : \ p \in \mathcal{U}\}, \ \ H(\mathcal{U}) = \{H(p) : \ p \in \mathcal{U}\}, \ \text{and} \ \ D_c(\mathcal{U}) = \{D_c(p) : \ p \in \mathcal{U}\}$$

are also compact. Since the deviations between the sets (4.5) and (4.6) have extremal polynomials.

## 4.1 Deviation of $A(p)$ from $A(p')$

The estimate of the deviation of $A(p)$ from $A(p')$ is the contains of Conjecture 1, which may be formulated in the following form.

• *For every $n \geq 2$, the equality*

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}'_n)) := \sup\{\rho(A(p); A(p')) : \ p \in \mathcal{P}_n\} = 1$$

*holds.*

There are many proofs of Conjecture 1 for polynomials of degree 3, see [3, 16, 18, 14, 11, 17, 9, 2, 21] and of degree 4, see [16, 18, 14, 11, 8, 2].

Conjecture 1 was proved for $n \leq 5$ by Meir, A. and A. Sharma [14] in 1969, see also [12, 2]. There was a brake for more than 20 years before the proof for $n \leq 6$ was published by Brown, J. [4] in 1991, see also [1, 2]. The case $n = 7$ was proved first by Borcea, J. [2] in 1996 and by Brown, J. [6] in 1997. In a recent paper J. E. Brown and G. Xiang (1999) [7] proved Conjecture 1 for $n \leq 8$. The proof is very elaborate and is based on obtaining good upper and lower estimates on the product of the moduli of the critical points of $p$. The next step to $n = 9$ is possible by this method through very scrupulous calculations.

After every proof of Conjecture 1 for a given $n$, it usually follows a proof for every polynomial with $n$ distinct zeros. It is natural to formulate

**Conjecture 3** *Conjecture 1 is true for every polynomial of the form* $q(z) = (z - z_1)^{m_1}(z - z_2)^{m_n} \cdots (z - z_n)^{m_n}$, *if it is true for* $p(x) = (z - z_1)(z - z_2) \cdots (z - z_n)$.

In 1969 G. Schmeisser [18] proved that Conjecture 1 is true for every polynomial $p \in \mathcal{P}_n$ with $p(0) = 0$.

Let $p \in \mathcal{P}_n$ and have the form

$$p(z) = \sum_{k=0}^{m} a_k z^{n_k}; \quad n = n_m > n_{m-1} > \cdots > n_0. \tag{4.8}$$

If $m < n$, the polynomial (4.8) is lacunary.

**Theorem 4.1** (G. Schmeisser [19, Theorem 2]). *Conjecture 1 is true for the polynomial* (4.8), *if* $n \geq 3m - 2$.

**Theorem 4.2** (G. Schmeisser [19, Theorem 3]). *Conjecture 1 is true for* $p \in \mathcal{P}_n$, *if*

$$p(z) = z^n + a_2 z^{n_2} + a_1 z^{n_1} + a_0 z^{n_0}, \quad \text{where} \quad n > n_2 > n_1 > n_0.$$

D. Phelps and R. S. Rodriguez (1972) [15] proved Conjecture 1 for all polynomials $p$ with $H(p)$ a segment. If for a polynomial $p \in \mathcal{P}(1)$, the convex hull $H(p)$ is a segment, then $p$ has only real zeros and real coefficients.

In 1977 G. Schmeisser [19] proved Conjecture 1 for polynomials $p$ with a convex hull $H(p)$, which is a triangle. The proof is purely geometric.

**Conjecture 4** (Schmeisser, G [19]). *Conjecture 1 is true for all* $p \in \mathcal{P}$ *for which* $H(p)$ *is a quadrangular.*

## 4.2 Deviation of $A(p)$ from $A(p'')$

It is natural to expect that the problem to find $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n''))$ is much more difficult than to find $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n'))$.

From Theorem 3.1 we have that $\overline{\rho}(A(\mathcal{P}_4); A(\mathcal{P}_4'')) \leq 4/3$.

**Conjecture 5** $\overline{\rho}(A(\mathcal{P}_4); A(\mathcal{P}_4'')) = 2/\sqrt{3}$.

This conjecture is motivated by the polynomial $p(z) = (z^2 + \frac{2}{3}z + 1)(z - 1)^2$, for which $\rho(A(p); A(p'')) = 2/\sqrt{3}$.

For the polynomial $z^5 - 5z + 4$ we have $\rho(A(p); A(p'') \geq 1.06 r_c(p)$, hence

$$\overline{\rho}(A(\mathcal{P}_5); A(\mathcal{P}_5'')) \geq 1.06 \ .$$

### 4.3 Deviation of $A(p)$ from $A(p^{(n-2)})$ for a polynomial $p \in \mathcal{P}_n$

**Lemma 4.1** *For $n \geq 5$, the inequalities*

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(n-2)})) \geq \begin{cases} 2\frac{n-1}{n+1} & \text{for } n \text{ odd,} \\ 1 + \sqrt{\frac{(n-2)(n-4}{n(n+2)}} & \text{for } n \text{ even} \end{cases}$$

*hold.*

**Proof.** For $n = 2m+1$, consider the polynomial

$$p(z) = \left(z^2 + \frac{2m}{m+1}z + 1\right)^m (z-1)$$

with

$$p^{(2m-1)}(z) = \frac{(2m+1)!}{2}\left(z + \frac{m-1}{m+1}\right)^2.$$

$p^{(2m-1)}$ has a double zero $\zeta_1 = \zeta_2 = -(m-1)/(m+1)$ and

$$\rho(A(p); A(p^{(n-2)})) = |1 - \zeta_1| = \frac{2m}{m+1} = 2\frac{n-1}{n+1}.$$

For $n = 2m+2$, consider the polynomial

$$p(z) = \left(z^2 + 2\sqrt{\frac{m^2-1}{m(m+2)}}z + 1\right)^m (z^2 - 1)$$

with

$$p^{(2m)}(z) = \frac{(2m+2)!}{2}\left(z + \sqrt{\frac{m(m-1)}{(m+1)(m+2)}}\right)^2.$$

$p^{(2m)}$ has a double zero $\zeta_1 = \zeta_2 = -\sqrt{m(m-1)/(m+1)(m+2)}$ and

$$\rho(A(p); A(p^{(n-2)})) = |1 - \zeta_1| = 1 + \sqrt{\frac{(n-2)(n-4)}{n(n+2)}}. \qquad \square$$

**Conjecture 6** *If $p \in \mathcal{P}_n$ is an extremal polynomia, for $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(s)}))$, then*

$$P^{(s)}(z) = \frac{n!}{(n-s)!}(z - \zeta_1)^{n-s}.$$

## 4.4 Deviation of $A(p)$ from $H(p')$ and $D_c(p')$

From (4.7) and Lemma 2.1 there follows that

$$\rho(A(p);A(p')) \geq \rho(A(p);H(p')) \geq \rho(A(p);D_c(p')). \tag{4.9}$$

Then we formulate a weaker conjecture than Conjecture 1.

**Conjecture 7** *For every $n \geq 2$, the following equalities hold:*

$$\overline{\rho}(A(\mathcal{P}_n);D_c(\mathcal{P}'_n)) = 1 \quad \text{and} \quad \overline{\rho}(A(\mathcal{P}_n);H(\mathcal{P}'_n)) = 1. \tag{4.10}$$

The equalities (4.10) are motivated by (4.9) and by the polynomial $p(z) = z^n - 1 \in \mathcal{P}_n$.

## 4.5 Deviations of $H(p)$

In 1977, G. Schmeisser formulated

**Conjecture 8** (Schmeisser's Conjecture [19]). *For every $n \geq 2$, the equality $\overline{\rho}(H(\mathcal{P}_n);A(\mathcal{P}'_n)) = 1$ holds.*

We conjecture also that

$$\overline{\rho}(H(\mathcal{P}_n);D_c(\mathcal{P}'_n)) = 1 \quad \text{and} \quad \overline{\rho}(H(\mathcal{P}_n);H(\mathcal{P}'_n)) = 1$$

for every $n \geq 2$.

## 4.6 Deviations of $D$

It is obvious that $\overline{\rho}(D;A(\mathcal{P}'_2)) = 1$.

**Conjecture 9** *For every $n \geq 3$, the equality*

$$\overline{\rho}(D;A(\mathcal{P}'_n)) = \sqrt{2} \tag{4.11}$$

*holds.*

A motivation for Conjecture 9 is the polynomial

$$p(z) = (z^2+1)(z-1)^{n-2} \in \mathcal{P}_n. \tag{4.12}$$

The zeros of the derivative $p'(z) = (nz^2 - 2z + n - 2)(z+1)^{n-3}$ are

$$\zeta_1 = (1 + i\sqrt{n^2-2n-1})/n, \quad \zeta_2 = (1 - i\sqrt{n^2-2n-1})/n \tag{4.13}$$

and $\zeta_j = 1$ for $j = 3,4,\ldots,n-1$. A simple calculation gives $\rho(D;A(p')) \geq |1+\zeta_1| = \sqrt{2}$, hence

$$\overline{\rho}(D;A(\mathcal{P}'_n)) \geq \sqrt{2}. \tag{4.14}$$

**Theorem 4.3** *Conjectute 9 is true for $n = 3$.*

**Proof.** Let $\overline{p} \in \mathcal{P}_3$ be an extremal polynomial for $\overline{\rho}(D; A(\mathcal{P}_3'))$. By Corollary 2.2 we may suppose that $\overline{p}(z) = (z^2 - 2\alpha z + 1)(z - z_3)$, where $\alpha \geq 0$. Let $\overline{p}'(z) = 3(z-\zeta_1)(z-\zeta_2)$ and suppose that $\rho(1; H(\overline{p}')) > \sqrt{2}$. Then $|1-\zeta_1| > \sqrt{2}$, $|1-\zeta_2| > \sqrt{2}$ and

$$|\overline{p}(1)| = 3|1 - \zeta_1||1 - \zeta_2| > 6.$$

On the other hand

$$|\overline{p}'(1)| = |3 - 2(2\alpha + z_3) + 1 + 2\alpha z_3| = 2|1 - \alpha||2 - z_3| \leq 6. \qquad \square$$

The Hausdorff deviation $\rho(D; H(p'))$ is larger than 1 for some polynomials $p \in \mathcal{P}$.

**Conjecture 10** *For every $n \geq 3$, the equality*

$$\overline{\rho}(D; H(\mathcal{P}_n')) = 1 + \frac{1}{n} \tag{4.15}$$

*holds.*

A motivation for Conjecture 10 is also the polynomial (4.12), from which we have that

$$\overline{\rho}(D; H(\mathcal{P}_n')) \geq 1 + \frac{1}{n}.$$

**Theorem 4.4** *Conjectute 10 is true for $n = 3$.*

**Proof.** As in the proof of Theorem 4.3, let $\overline{p} \in \mathcal{P}_3$ be an extremal polynomial for $\overline{\rho}(D; H(\mathcal{P}_3'))$ and $\overline{p}(z) = (z^2-2\alpha z+1)(z-z_3)$, where $\alpha \geq 0$. Let $\overline{p}'(z) = 3(z-\zeta_1)(z-\zeta_2)$ and suppose that $\rho(1; H(\overline{p}')) > 4/3$. Then

$$\left|1 - \frac{\zeta_1 + \zeta_2}{2}\right| > \frac{4}{3}.$$

On the other hand

$$\overline{p}'(z) = 3z^2 - 2(2\alpha + z_3)z + 1 + 2\alpha z_3,$$

hence $\zeta_1 + \zeta_2 = \frac{2}{3}(2\alpha + z_3)$ and

$$\left|1 - \frac{1}{3}(2\alpha + z_3)\right| = \frac{1}{3}|2(1 - \alpha) + (1 - z_3)| \leq \frac{4}{3}. \qquad \square$$

For the Hausdorff deviation $\rho(D; D_c(p'))$ we formulate

**Conjecture 11** *For every $n \geq 3$, the equality $\overline{\rho}(D; D_c(\mathcal{P}_n')) = 1$ holds.*

**Theorem 4.5** *Conjectute 11 is true for $n = 3$.*

**Proof.** As in the proof of Theorem 4.3 and Theorem 4.4, let $\overline{p} \in \mathcal{P}_3$ be an extremal polynomial for $\overline{\rho}(D; D(\mathcal{P}_3'))$ and $\overline{p}(z) = (z^2 - 2\alpha z + 1)(z - z_3)$, where $\alpha \geq 0$. Let $\overline{p}'(z) = 3(z - \zeta_1)(z - \zeta_2)$ and suppose that $\rho(1; D(\overline{p}')) > 1$. Then

$$\left|1 - \frac{\zeta_1 + \zeta_2}{2}\right| - \left|\frac{\zeta_1 + \zeta_2}{2}\right| > 1$$

and

$$\left|1 - \frac{1}{3}(2\alpha + z_3)\right| - \frac{1}{3}|2\alpha + z_3)| > 1.$$

On the other hand

$$\left|1 - \frac{1}{3}(2\alpha + z_3)\right| \leq 1 + \frac{1}{3}|2\alpha + z_3)|,$$

which is contradiction. $\square$

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