On the zeros of an algebraic polynomial and its consecutive derivatives

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1 Introduction

In 1958 we formulated the following [2, 5]:

Conjecture 1 *If all the zeros of the polynomial* $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ $(n \geq 2)$ *lie in the unit disk* $D = D(0,1) = \{z : \ |z| \leq 1\}$, *then for every* z_k *the disk* $D(z_k, 1) = \{z : \ |z - z_k| \leq 1\}$ *contains at least one zero of* $p'(z)$.

In this paper we formulate and investigate a generalization of this conjecture.

Conjecture 2 *If all the zeros of the polynomial* $p(z)$ *lie in the unit disk, then for every zero* z_k *of* p, *the disk*

$$D\left(z_k, \frac{2s}{s+1}\right) = \left\{z : \ |z - z_k| \leq \frac{2s}{s+1}\right\}$$

contains at least one zero of $p^{(s)}(z)$; $s = 1, 2, \ldots, n-1$.

2 Preliminaries and notations

As we are interested only in the relative location of the zeros of a polynomial and the zeros of its derivatives, it is sufficient to consider only monic polynomials $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. The geometric interrelations between the set $A(p)$ of zeros of p and the set $A(p^{(s)})$ of zeros of $p^{(s)}$ are invariant with respect to rotation, translation and scaling. Studying these interrelations, we may prescribe the center and the radius of the smallest disk, which contains the set $A(p)$.

Set the following notations:

1. $A(p)$ - the set of all distinct zeros of the polynomial p;
2. $H(p) = H(A(p))$ - the **convex hull** of p;
3. $\mathcal{P}_n$ - the set of all monic polynomials p of degree $n \geq 2$, such that the smallest disk containing $A(p)$ is the unit disk;

Lemma 2.1 *If $p \in \mathcal{P}_n$, then:*

1. *At least two zeros of p lie in the unit circle $C = \{z : |z| = 1\}$;*
2. *At least one zero of p lies in each subarc of C with length π.*

First we list two classical theorems from the Geometry of polynomials. For more results see [3].

Theorem 2.1 (Gauss–Lucas). *For every algebraic polynomial p, $A(p') \subset H(p)$.*

Definition 2.1 *The polynomials*

$$p(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are called **apolar** *if*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{2.1}$$

The equation (2.1) *is called the* **apolarity condition**.

Theorem 2.2 (Grace Apolarity Theorem). *Let p and q be apolar, then any circular region containing all zeros of p or q contains at least a zero of the other.*

A circular region is a closed disk or a closed half-plain.

Lemma 2.2 *If $p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$ and $a_0 \neq 0$, then p has at least one zero in the disk $D(0, r)$, where $r = |a_m/a_0|^{1/m}$.*

Proof. The polynomial $q(z) = a_0 z^m - (-1)^m a_m$ is apolar to the polynomial p and all of the zeros of q lie in the disk $D(0, |a_m/a_0|^{1/m})$. The proof of the lemma follows from the Grace Apolarity Theorem 2.2. $\square$

Lemma 2.3 *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|p^{(s)}(z_1)| \leq (n!/(n-s)!) r^{n-s}$, $s = 1, 2, \ldots, n-1$, then there exists a $\zeta_1 \in A(p^{(s)})$, such that $|z_1 - \zeta_1| \leq r$.*

Proof. We have

$$p^{(s)}(z_1 - z) = (-1)^{n-s} \frac{n!}{(n-s)!} z^{n-s} + \cdots + p^{(s)}(z_1).$$

By Lemma 2.2

$$\left| \frac{a_{n-s}}{a_0} \right| = \left| \frac{p^{(s)}(z_1)(n-s)!}{n!} \right| = r^{n-s},$$

or $r = |a_m/a_0|^{1/(n-s)}$. $\square$

Lemma 2.4 *For every $p \in \mathcal{P}_n$ and $s \in \{1, 2, \ldots, n-1\}$, the following inequality*

$$|p^{(s)}(z_1)| \le s! \binom{n-1}{n-s} (1 + |z_1|)^{n-s}$$

holds.

Proof. From $p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$ and $|z_k| \le 1;\ \ k = 1, 2, \ldots, n$ there follows

$$|p^{(s)}(z_1)| \le s! \sum_{2 \le k_1 \le \cdots \le k_{n-s} \le n} |(z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}})|$$

$$\le s! \sum_{2 \le k_1 \le \cdots \le k_{n-s} \le n} (1 + |z_1|)^{n-s} = s! \binom{n-1}{n-s} (1 + |z_1|)^{n-s}. \qquad \square$$

From Lemma 2.3 and Lemma 2.4 follows a rough estimate:

Theorem 2.3 *If $p \in \mathcal{P}_n$, then for every zero z_k of p there exists a zero ζ_l of $p^{(s)}$, such that*

$$|z_k - \zeta_l| \le 2 \left(\frac{s}{n}\right)^{\frac{1}{n-s}};\ \ s = 1, 2, \ldots n-1.$$

Proof. From Lemma 2.4, for every zero z_k of p

$$|p^{(s)}(z_k)| \le s! \binom{n-1}{n-s} 2^{n-s}.$$

Hence, from Lemma 2.3, we have

$$\frac{n!}{(n-s)!} r^{n-s} = s! \binom{n-1}{n-s} 2^{n-s}, \quad \text{or} \quad r = 2 \left(\frac{s}{n}\right)^{\frac{1}{n-s}}. \qquad \square$$

Theorem 2.3 is not trivial, as from the Gauss – Lucas Theorem follows only that $|z_k - \zeta_l| \le 2$.

3 Generalized conjecture

In this section we consider partial proofs of Conjecture 2.

For $n = 1$, Conjecture 2 is Conjecture 1. For $n = n - 1$, Conjecture 2 is trivial, as $p^{(n-1)}(z)$ has only one zero $\zeta_1 = (z_1 + z_2 + \cdots + z_n)/n$. Then

$$|z_1 - \zeta_1| \le \sup \left\{ z_1 - \frac{z_1 + z_2 + \cdots + z_n}{n} :\ |z_k| \le 1, k = 1, 2, \ldots, n \right\} = \frac{2(n-1)}{n}.$$

Lemma 3.1 *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|z_1| = 1$, then ther exists a zero ζ_1 of $p^{(s)}(z)$ on the disk $D(z_1/(s+1), s/(s+1)) \subset D(z_1, 2s/(s+1))$.*

Proof. Using the method of A. Meir and A. Sharma [4], we may set $z_1 = 1$ and $p(z) = (z-1)q(z)$. Let $\zeta_1, \zeta_2, \dots, \zeta_{n-s}$ be the zeros of $p^{(s)}$ and $\eta_1, \eta_2, \dots, \eta_{n-s}$ be the zeros of $q^{(s-1)}$. By Gauss–Lucas Theorem 2.1 $|\eta_k| \le 1$; $k = 1, 2, \dots, n-s$. If $p^{(s)}(1) = 0$, we are done. Let $p^{(s)}(1) \ne 0$, then

$$\frac{p^{(s+1)}(1)}{p^{(s)}(1)} = \frac{s+1}{s} \cdot \frac{q^{(s)}(1)}{q^{(s-1)}(1)}$$

and

$$\sum_{j=1}^{n-s} \Re\left(\frac{1}{1-\zeta_j}\right) = \frac{s+1}{s} \sum_{k=1}^{n-s} \Re\left(\frac{1}{1-\eta_k}\right) \ge \frac{(s+1)(n-s)}{2s}.$$

If $\Re(1/(1-\zeta_1)) = \max\{\Re(1/(1-\zeta_j)) :\ j = 1, 2, \dots, n-s\}$, then

$$\Re\left(\frac{1}{1-\zeta_1}\right) \ge \frac{s+1}{2s},$$

therefore

$$\zeta_1 \in D(1/(s+1), s/(s+1)) \subset D(1, 2s/(s+1)). \qquad \square$$

Corollary 3.1 *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|z_1| = 1$, then there exists $\zeta_1 \in A(p^{(s)})$, such that*

$$|z_1 - \zeta_1| \le \frac{2s}{s+1}; \quad s = 1, 2, \dots, n-1.$$

Corollary 3.2 *Conjecture 2 is true for a polynomial $p \in \mathcal{P}_n$ if all the corners of $H(p)$ are on the unit circle $C = \{z :\ |z| = 1\}$.*

Proof. If all the corners $z_{k_1}, z_{k_2}, \dots, z_{k_l}$ of $H(p)$ are on the unit circle, then

$$H(p) \subset \bigcup_{m=1}^{l} D(z_{k_m}/(s+1), s/(s+1)).$$

If a zero $z_1 \in D(z_{k_1}/(s+1), s/(s+1)) \subset D(z_1, 2s/(s+1))$, then there exists a zero ζ_{k_1} of $p^{(s)}$, such that $|z_1 - \zeta_{k_1}| \le 2s/(s+1)$. $\square$

Lemma 3.2 *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$, $|z_1| < 1$ and $s \in \{1, 2, \dots, n-1\}$, then*

$$|p^{(s)}(z_1)| \le \frac{(n-3)!}{(n-s)!} 2^{n-s-1} s(n^2 + s^2 - 3n - 3s + 4).$$

Proof. As $|z_1| < 1$, by Lemma 2.1, there are two zeros $z_2, z_3 \in A(p)$ such that $|z_1| = |z_2| = 1$,

$$\arg z_2 \le \arg z_1 \le \arg z_3 \quad \text{and} \quad 0 \le \arg z_3 - \arg z_2 \le \pi.$$

Then, it is geometrically obvious that

$$|z_1 - z_2||z_1 - z_3| \le 2 \quad \text{and} \quad \left| z_1 - \frac{z_2 + z_3}{2} \right| \le 1. \tag{3.2}$$

Using (3.2), we have, as in Lemma 2.4

$$p^{(s)}(z_1) = s! \sum_{2 \le k_1 \le \cdots \le k_{n-s} \le n} (z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}}) =$$

$$s! \left[(z_1 - z_2)(z_1 - z_3) \sum_{4 \le k_3 \le \cdots \le k_{n-s} \le n} (z_1 - z_{k_3})(z_1 - z_{k_4}) \cdots (z_1 - z_{k_{n-s}}) + \right.$$

$$2 \left(z_1 - \frac{z_2 + z_3}{2} \right) \sum_{4 \le k_2 \le \cdots \le k_{n-s} \le n} (z_1 - z_{k_2})(z_1 - z_{k_3}) \cdots (z_1 - z_{k_{n-s}}) +$$

$$\left. \sum_{4 \le k_1 \le \cdots \le k_{n-s} \le n} (z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}}) \right]$$

and

$$|p^{(s)}(z_1)| \le s! \left[2 \cdot 2^{n-s-2} \binom{n-3}{n-s-2} + 2 \cdot 2^{n-s-1} \binom{n-3}{n-s-1} + 2^{n-s} \binom{n-3}{n-s} \right] =$$

$$\frac{(n-3)!}{(n-s)!} 2^{n-s-1} s(n^2 + s^2 - 3n - 3s + 4). \qquad \square$$

Theorem 3.1 *Conjecture 2 is true for:*
1) $s = 2$ and $n = 4, 5, 6$;
2) $s = n - 2$ and arbitrary n;
2) $s = n - 3$ and arbitrary n.

Proof From Corollary 3.1 and Lemma 3.2 follows that Conjecture 2 is true for a given pair (s, n) if

$$s \frac{n^2 + s^2 - 3n - 3s + 4}{n(n-1)(n-2)} \le 2 \left(\frac{s}{s+1} \right)^{n-s}. \tag{3.3}$$

For $s = 1$, the inequality (3.3) is $n \ge 2^{n-2}$, and that is an easy proof of Conjecture 1 fore $n = 3, 4$. Conjecture 1 is proved for $n \le 8$, see [1].

For $s = 2$, the inequality (3.3) is $n \geq (3/2)^{n-2}$, therefore Conjecture 2 is true for the pears (2,4), (2,5) and (2,6).

For $s = n - 2$ the inequality (3.3) is $n - 2 \leq n - 1$, hence Conjecture 2 true for every n and $s = n - 2$.

For $s = n - 3$ the inequality (3.3) is

$$\frac{n^2 - 6n + 11}{n(n-1)} \leq \left(\frac{n-3}{n-2}\right)^2,$$

which is satisfied for every $n \geq 4$. Therefore Conjecture 2 true for every n and $s = n - 3$. $\square$

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