On the Hausdorff Geometry of Polynomials

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1 Introduction

A set of n points $A = \{z_1, z_2, \ldots, z_n\}$ on the complex plane defins the monic polynomial $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$ with derivative $p'(z) = n(z - \zeta_1)(z - \zeta_2)\cdots(z - \zeta_{n-1})$. The Geometry of polynomials studies the relations between the sets A and $A' = \{\zeta_1, \zeta_2, \ldots, \zeta_{n-1}\}$. The classical Gauss – Lucas Theorem asserts that the convex hull of A contains A'.

In 1958 I formulated the following conjecture in the Geometry of Polynomials [19, 39].

Conjecture 1 *If all the zeros of the polynomial* $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ $(n \geq 2)$ *lie in the unit disk* $D = D(0,1) = \{z : \ |z| \leq 1\}$, *then for every* z_k *the disk* $D(z_k, 1) = \{z : \ |z - z_k| \leq 1\}$ *contains at least one zero of* $p'(z)$.

Up to now, Conjecture 1 was neither proved nor disproved, although more than 80 related papers have been published on this topic. In attempts to prove this conjecture, many new results have been obtained and many other problems and conjectures have been formulated [39]. Our purpose in this paper is to review some of these results and to formulate some new problems from the point of view of Hausdorff distance between the set of zeros and the set of critical points of a polynomial.

2 Preliminaries and notations

Let A and B be two bounded and closed sets of points in the complex plane $\mathcal{C}$. We will use the notations:

1. $\rho(b; A) = \inf\{|b - a| : \ a \in A\}$,
2. $\rho(B; A) = \sup\{\rho(b; A) : \ b \in B\}$ - the **Hausdorff deviation** of B from A,
3. $h(A, B) = \max\{\rho(A; B), \rho(B; A)\}$ - the **Hausdorff distance** between A and B.
4. $H(A)$ - the smallest closed and convex point set, which contains A, called the **convex hull** of A.

5. $D_c(A) = D(c_c(A), r_c(A))$ - the **circumself disk** of A, i. e., the smallest closed disk, which contains A, with $c_c(A)$ - the **circumself center** of A and $r_c(A)$ - the **circumself radius** of A;

6. $C_c(A) = \{z : \ |z - c_c(A)| = r_c(A)\}$ - the **circumself circle**[1] of A;

The following relations are obvious for every $A \subset \mathcal{C}$:

$$A \subset H(A) \subset D_c(A). \tag{2.1}$$

A simple and useful statement is

Lemma 2.1 *If A, B, V are bounded point sets in $\mathcal{C}$ and $B \subset V$, then*

$$\rho(A; B) \geq \rho(A; V) \quad \text{and} \quad \rho(B; A) \leq \rho(V; A).$$

Proof. From the definition

$$\rho(A; B) = \sup\{\rho(a; B) : \ a \in A\} \geq \sup\{\rho(a; V) : \ a \in A\} = \rho(A; V),$$

as $B \subset V$ and

$$\rho(a; B) = \inf\{|a - b| : \ b \in B\} \geq \inf\{|a - v| : \ v \in V\} = \rho(a; V).$$

In the same way

$$\rho(V; A) = \sup\{\rho(v; A) : \ v \in V\} \geq \sup\{\rho(b; A) : \ b \in B\} = \rho(B; A),$$

as $B \subset V$. $\square$

In general $\rho(B; A) \neq \rho(A; B)$.

Corollary 2.1 *If A and B are two bounded and closed point sets in the complex plane. Then $h(H(A), H(B)) \leq h(A, B)$.*

From the definition of circumself disk, there follows

Lemma 2.2 *If $A \subset \mathcal{C}$ is a closed and bounded (and has at least two distinct points), then:*

1. *At least two distinct points of A lie in $C_c(A)$;*

2. *At least one point of A lies in each subarc of $C_c(A)$ with length π.*

Lemma 2.3 *Let $A \subset D$, $b \notin H(A)$ and $\rho(b; H(A)) \leq 1$. Then $\rho(b; A) \leq \sqrt{2}$.*

Proof. Let a be the unique point for which $|a - b| = \rho(b; H(A))$. Observe that $H(A)$ is convex. Let l be the support line of $H(A)$, which separates b from $H(A)$ and, which is orthogonal to the line segment $[a, b]$. If $a \in A$, then $\rho(b; A) = \rho(b; H(A)) \leq 1$. If $a \notin A$, then there are at least two points $a_1, a_2 \in A$, which lie on l and a lies in the segment $[a_1, a_2]$. Let $|a - a_1| \leq |a - a_2|$, then $|a - a_1| \leq 1$, as $|a_1 - a_2| \leq 2$. Then $|b - a_1| = \sqrt{|b - a|^2 + |a - a_1|^2} \leq \sqrt{2}$. $\square$

[1]Note the difference between circumcircle of a triangle and circumself circle

2.1 Polynomial sets

As we are interested only in the relative location of the zeros and the critical points of a polynomial, it is sufficient to consider only monic polynomials $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. The geometric interrelations between the set $A(p)$ of zeros and the set $A(p')$ of critical points of a polynomial p are invariant with respect to rotation, translation and scaling. Studying these interrelations, we may prescribe the circumself center and the circumself radius of the set $A(p)$.

Definition 2.1 *Set the following notations:*

1. $A(p)$ *- the set of all distinct zeros of the polynomial* p*;*

2. $H(p) = H(A(p))$ *- the* **convex hull** *of* p*;*

3. $D_c(p) = D_c(A(p)) = D(c_c(p), r_c(p))$ *- the* **circumself disk** *of* p *with* $c_c(p) = c_c(A(p))$ *- the* **circumself center** *of* p *and* $r_c(p) = r_c(A(p))$ *- the* **circumself radius** *of* p*;*

4. $\mathcal{P}$ *- the set of all monic polynomials* p *of degree* $n \geq 2$*, with circumself disk* $D_c(p) = D(c_c(p), r_c(p)) = D(0, 1) = D$*;*

5. $\mathcal{P}_n(a)$ *- the set of all polynomials* $p \in \mathcal{P}$ *with* $p(a) = 0$*;*

6. $\mathcal{P}_{n,m}$*;* $m \leq n$ *- the set of all polynomials* $p \in \mathcal{P}_n$ *of the form* $p(z) = \prod_{k=1}^{m}(z - z_k)^{n_k}$*, where* $n = n_1 + n_2 + \cdots + n_m$ *and* $z_1, z_2, \ldots, z_m$ *are distinct.*

7. $c_g(p) = (n_1 z_1 + n_2 z_2 + \cdots + n_m z_m)/n$ *- the* **gravity center** *of* $p \in \mathcal{P}_{n,m}$*;*

Lemma 2.4 *If* $p \in \mathcal{P}$*, then* $c_c(p') \in H(p) \subset D$*.*

Proof. On the contrary, suppose that $c_c(p') \notin H(p)$. Let l be a support line of $H(p)$, which separates $H(p)$ and $c_c(p')$. Let L be the half-plane defined by l, which contains $H(p)$. Then the subarc λ of the circle $C(c_c(p'), r_c(p')) = C_c(p')$ outside L has length larger than π. By Lemma 2.2, at least one zero of p' lies in λ, which is outside $H(p)$. This contradicts the Gauss–Lucas Theorem. □

From Lemma 2.2 there follows:

Corollary 2.2 *If* $p \in \mathcal{P}_{n,m}$*, then:*

1. *At least two zeros of* p *lie in* C*;*

2. *At least one zero of* p *lies in each subarc of* C *with length* π*, therefore* $m \geq 2$*.*

Remark 2.1 *If* $p, q \in \mathcal{P}_n$ *and* $h(A(p); A(q)) = 0$*, there follows only that* $A(p) = A(q)$*. If* p *and* q *have multiple zeros, then we may have* $p \neq q$*.*

2.2 Basic statements

First we list some classical theorems from the Geometry of polynomials. For more results see [23].

Theorem 2.1 (Gauss–Lucas). *For every algebraic polynomial p, $A(p') \subset H(p)$.*

Theorem 2.2 (Grace–Heawood). *If z_1 and z_2 are any two distinct zeros of a polynomial p of degree n, then at least one critical point of p lies in the disk $D((z_1 + z_2)/2, (|z_1 - z_2|/2)\cot(\pi/n))$.*

Theorem 2.3 *Let z_1 and z_2 be two different zeros of the polynomial p and l be the bisector of the segment from z_1 to z_2. Then in every two closed half-plane, defined by l, there is at least one zero of p'.*

The bisector property was noticed for the first time by G. Szegó [40].

Corollary 2.3 *If $p(z)$ is a polynomial and $p(1) = p(-1)$, then $p'(z)$ has at least one zero in each half-plane $\{z : \mathrm{Re} z \geq 0\}$ and $\{z : \mathrm{Re} z \leq 0\}$.*

Definition 2.2 *The polynomials*

$$p(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are called **apolar** *if*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{2.2}$$

The equation (2.2) *is called the* **apolarity condition**.

Theorem 2.4 (Grace Apolarity Theorem). *Let p and q be apolar, then any circular region containing all zeros of p or q contains at least a zero of the other.*

A circular region is a closed disk or a closed half-plain.

Theorem 2.5 (Walsh). *If a polynomial p of degree n has m_1 zeros in a circle $C_1 = C(c_1, r_1)$ and the remaining $m_2 = n - m_1$ zeros in a circle $C_2 = C(c_2, r_2)$, then every critical point of p that does not lie in C_1 or in C_2 lies in a third circle $C_3 = C((m_2 c_1 + m_1 c_2)/n, (m_2 r_1 + m_1 r_2)/n)$.*

Lemma 2.5 *If $p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$ and $a_0 \neq 0$, then p has at least one zero in the disk $D(0, r)$, where $r = |a_m/a_0|^{1/m}$.*

Proof. The polynomial $q(z) = a_0 z^m - (-1)^m a_m$ is apolar to the polynomial p and all of the zeros of q lie in the disk $D(0, |a_m/a_0|^{1/m})$. The proof of the lemma follows from the Grace Apolarity Theorem 2.4. $\square$

Lemma 2.6 *If* $p \in \mathcal{P}_n(a)$ *and* $|p^{(s)}(a)| \leq (n!/(n-s)!)r^{n-s}$, $s = 1, 2, \ldots, n-1$, *then* $\rho(a; A(p^{(s)})) \leq r$.

Proof. We have

$$p^{(s)}(a-z) = (-1)^{n-s}\frac{n!}{(n-s)!}z^{n-s} + \cdots + p^{(s)}(a).$$

By Lemma 2.5

$$\left|\frac{a_{n-s}}{a_0}\right| = \left|\frac{p^{(s)}(a)(n-s)!}{n!}\right| = r^{n-s},$$

or $r = |a_m/a_0|^{1/(n-s)}$. $\square$

Corollary 2.4 *If* $p \in \mathcal{P}_n(a)$ *and* $|p'(a)| \leq n$, *then* $\rho(a; A(p')) \leq 1$. *Also, if* $|p'(a)| < n$, *then* $\rho(a; A(p')) < 1$, *see* Z. Rubinstein *[34, Theorem 3].*

Lemma 2.7 *For every* $p \in \mathcal{P}_n$ *and* $s \in \{1, 2, \ldots, n-1\}$, *the following inequality*

$$|p^{(s)}(z_1)| \leq s!\binom{n-1}{n-s}(1+|z_1|)^{n-s}$$

holds.

Proof. From $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ and $|z_k| \leq 1;\ \ k = 1, 2, \ldots, n$ there follows

$$|p^{(s)}(z_1)| \leq s! \sum_{2\leq k_1\leq\cdots\leq k_{n-s}\leq n} |(z_1-z_{k_1})(z_1-z_{k_2})\cdots(z_1-z_{k_{n-s}})|$$

$$\leq s! \sum_{2\leq k_2\leq\cdots\leq k_{n-s}\leq n}(1+|z_1|)^{n-s} = s!\binom{n-1}{n-s}(1+|z_1|)^{n-s}. \quad \square$$

A. Meir and A. Sharma [27] considered the deviation of the zeros of a special polynomial from the zeros of its s^{th} derivative.

Theorem 2.6 (A. Meir and A. Sharma [27]). *Let* $p \in \mathcal{P}_n(1)$ *and let the zero 1 be of multiplicity* ν. *Then at least one zero of* $p^{(s)}(z)$, $1 \leq s \leq n-1$, *lies in the disk* $D(\nu/(s+1), 1-\nu/(s+1))$.

Corollary 2.5 (A. G. Goodman, Q. I. Rahman and J. Ratti [18], G. Schmeisser [36]). *If* $p \in \mathcal{P}(1)$, *then there exists* $\zeta_1 \in A'(p)$ *such that* $\zeta_1 \in D(1/2, 1/2)$.

We will use the following corollary of Theorem 2.6.

Corollary 2.6 *If* $p \in \mathcal{P}_n(1)$, $s = 1, 2, \ldots, n-1$, *then* $\rho(1; A(p^{(s)})) \leq 2s/(s+1))$.

Proof. Let $p(z) = (z-1)q(z)$, $\zeta_1, \zeta_2, \ldots, \zeta_{n-s}$ be the zeros of $p^{(s)}$ and $\eta_1, \eta_2, \ldots, \eta_{n-s}$ be the zeros of $q^{(s-1)}$. By Gauss–Lucas Theorem 2.1 $|\eta_k| \leq 1;\ k = 1, 2, \ldots, n-s$. If $p^{(s)}(1) = 0$, we are done. Let $p^{(s)}(1) \neq 0$, then

$$\frac{p^{(s+1)}(1)}{p^{(s)}(1)} = \frac{s+1}{s} \cdot \frac{q^{(s)}(1)}{q^{(s-1)}(1)}$$

and

$$\sum_{j=1}^{n-s} \Re\left(\frac{1}{1-\zeta_j}\right) = \frac{s+1}{s} \sum_{k=1}^{n-s} \Re\left(\frac{1}{1-\eta_k}\right) \geq \frac{(s+1)(n-s)}{2s}.$$

If $\Re(1/(1-\zeta_1)) = \max\{\Re(1/(1-\zeta_j)) :\ j = 1, 2, \ldots, n-s\}$, then

$$\Re\left(\frac{1}{1-\zeta_1}\right) \geq \frac{s+1}{2s},$$

hence

$$\zeta_1 \in D(1/(s+1), s/(s+1)) \subset D(1, 2s/(s+1)). \qquad \square$$

3 Hausdorff deviations

For a polynomial $p \in \mathcal{P}_n$, we shall consider the deviation of one of the sets

$$A(p), H(p), D_c(p) = D \tag{3.3}$$

from one of the sets

$$A(p^{(s)}), H(p^{(s)}), D_c(p^{(s)}), \tag{3.4}$$

for $s = 1, 2, \ldots n-1$.

By definition

$$A(p) \subset H(p) \subset D_c(p) = D \quad \text{and} \quad .A(p^{(s)}) \subset H(p^{(s)}) \subset D_c(p^{(s)}). \tag{3.5}$$

If $\mathcal{U}$ is a compact set of polynomials, then the sets

$$A(\mathcal{U}) = \{A(p) :\ p \in \mathcal{U}\},\quad H(\mathcal{U}) = \{H(p) :\ p \in \mathcal{U}\},\quad \text{and}\quad D_c(\mathcal{U}) = \{D_c(p) :\ p \in \mathcal{U}\}$$

are also compact. Since the deviations between the sets (3.3) and (3.4) have extremal polynomials.

3.1 Deviation of $A(p)$ from $A(p')$

The estimate of the deviation of $A(p)$ from $A(p')$ is the contains of Conjecture 1, which may be formulated in the following form.

Conjecture 1. *For every $n \geq 2$, the equality*

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}'_n)) := \sup\{\rho(A(p); A(p')) : \; p \in \mathcal{P}_n\} = 1 \tag{3.6}$$

holds.

There are many proofs of Conjecture 1 for polynomials of degree 3, see [7, 34, 36, 27, 20, 35, 14, 6, 43] and of degree 4, see [34, 36, 27, 20, 13, 6]. Here we present simple proofs of these cases. For polynomials of degree 2, Conjecture 1 is trivial.

Theorem 3.1 *Conjecture 1 is true for every polynomial $p \in \mathcal{P}_3(1)$.*

Proof. Let $p(z) = (z-1)(z-z_2)(z-z_3)$. If $|z_2| = |z_3| = 1$, then the proof follows from Corollary 2.5. If $|z_3| < 1$, then $z_2 = -1$ and

$$|p'(z_3)| = |(z_3 - 1)(z_3 + 1)| = |z_3^2 - 1| \leq 2.$$

Hence, the proof follows from Corollary 2.4 and Corollary 2.5. □

Theorem 3.2 *Conjecture 1 is true for every polynomial $p \in \mathcal{P}_4(1)$.*

Proof. Let $p(z) = (z-1)(z-z_2)(z-z_3)(z-z_4)$.

1) Let $|z_2| = |z_3| = |z_4| = 1$, then the proof follows from Corollary 2.5.

2) Let $|z_2| = |z_3| = 1$, $z_2 = e^{i\varphi_2}$, $z_3 = e^{i\varphi_3}$, $0 \leq \varphi_2 \leq \pi$, $\pi \leq \varphi_3 \leq 2\pi$ and $z_4 = r_4 e^{i\varphi_4}$. If $0 \leq \varphi_4 \leq \varphi_2$, then $|(z_4-1)(z_4-z_2)| \leq 2$, if $\varphi_2 \leq \varphi_4 \leq \varphi_3$, then $|(z_4-z_2)(z_4-z_3)| \leq 2$, if $\varphi_3 \leq \varphi_4 < 2\pi$, then $|(z_4-z_3)(z_4-1)| \leq 2$, hence

$$|p'(z_4)| = |(z_4-1)(z_4-z_2)||z_4-z_3| \leq 2 \cdot 2 = 4$$

and the proof follows from Corollary 2.4 and Corollary 2.5.

3) Let $|z_2| = 1$, $|z_3| < 1$ and $|z_4| < 1$, then $z_2 = -1$ and

$$|p'(z_3)| = |z_3^2 - 1||z_3 - z_4| \leq 2 \cdot 2 = 4,$$

$$|p'(z_4)| = |z_4^2 - 1||z_4 - z_3| \leq 2 \cdot 2 = 4.$$

Hence, the proof follows from Corollary 2.4 and Corollary 2.5. □

Conjecture 1 was proved for $n \leq 5$ by Meir, A. and A. Sharma [27] in 1969, see also [22, 6]. There was a brake for more than 20 years before the proof for $n \leq 6$ was published by Brown, J. [8] in 1991, see also [5, 6]. The case $n = 7$ was proved first by Borcea, J. [6] in 1996 and by Brown, J. [9] in 1997. In a recent paper J. E. Brown and G. Xiang (1999) [11] proved Conjecture 1 for $n \leq 8$. The proof is very elaborate and is based on obtaining good upper and lower estimates on the product of the moduli of the critical points of p. The next step to $n = 9$ is possible by this method through very scrupulous calculations.

After every proof of Conjecture 1 for a given n, it usually follows a proof for every polynomial with n distinct zeros. It is natural to formulate

Conjecture 2 *Conjecture 1 is true for every polynomial of the form* $q(z) = (z - z_1)^{m_1}(z - z_2)^{m_n} \cdots (z - z_n)^{m_n}$, *if it is true for* $p(x) = (z - z_1)(z - z_2) \cdots (z - z_n)$.

In 1969 G. Schmeisser [36] proved the following statement.

Theorem 3.3 *Conjecture 1 is true for every polynomial* $p \in \mathcal{P}(0)$.

Proof. An elegant geometric proof is given in G. Gacs (1971) [17], based on Theorem 2.3. Let $z_1 \neq 0$ be a zero of $p \in \mathcal{P}(0)$ and l be the bisector of the segment from z_1 to 0. Let L be the half-plane defined by l and $0 \notin L$. Then by Theorem 2.3 and Gauss–Lucas Theorem 2.1 it follows that $(L \bigcap D) \subset D(z_1, 1)$ contains at least one zero of p'. $\square$

Remark 3.1 *Conjecture 1 is trivially true for every* $p \in \mathcal{P}$, *if* $p'(0) = 0$.

Let $p \in \mathcal{P}_n$ and have the form

$$p(z) = \sum_{k=0}^{m} a_k z^{n_k}; \quad n = n_m > n_{m-1} > \cdots > n_0. \tag{3.7}$$

If $m < n$, the polynomial (3.7) is lacunary.

Theorem 3.4 (G. Schmeisser [38, Theorem 2]). *Conjecture 1 is true for the polynomial* (3.7), *if* $n \geq 3m - 2$.

Theorem 3.5 (G. Schmeisser [38, Theorem 3]). *Conjecture 1 is true for* $p \in \mathcal{P}_n$, *if*

$$p(z) = z^n + a_2 z^{n_2} + a_1 z^{n_1} + a_0 z^{n_0}, \quad \text{where} \quad n > n_2 > n_1 > n_0.$$

Theorem 3.6 *Conjecture 1 is true for a polynomial* $p \in \mathcal{P}$, *if all the corners of* $A(p)$ *lie in the unit circle.*

Proof. If all the corners of $A(p)$ lie in the unit circle, then by Corollary 2.5 for every zero $|z_k|$ of p with $|z_k| = 1$, the disk $D_k = D(z_k/2, 1/2)$ contains at least one zero of p'. If $\hat{A}(p) = \{z_1, z_2, \ldots, z_m\}$, $|z_k| = 1;\ k = 1, 2, \ldots, m$, are all the corners of $A(p)$, then $A(p) \subset \bigcup_{k=1}^{m} D_k$. $\square$

D. Phelps and R. S. Rodriguez (1972) [30] proved Conjecture 1 for all polynomials p with $H(p)$ a segment. If for a polynomial $p \in \mathcal{P}(1)$, the convex hull $H(p)$ is a segment, then p has only real zeros and real coefficients.

In 1977 G. Schmeisser [38] proved Conjecture 1 for polynomials p with a convex hull $H(p)$, which is a triangle. The proof is purely geometric.

Conjecture 3 (Schmeisser, G [38]). *Conjecture 1 is true for all* $p \in \mathcal{P}$ *for which* $H(p)$ *is a quadrangular.*

3.2 Deviation of $A(p)$ from $A(p'')$

It is natural to expect that the problem to find $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n''))$ is much more difficult than to find $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n'))$. Our first conjecture in this direction is the following.

Conjecture 4 *For $n \geq 5$, the equality $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n'')) = 1$ holds.*

If Conjecture 4 is not true, it will be possible to construct a contrexample by a computer. It is trivial that $\overline{\rho}(A(\mathcal{P}_3); A(\mathcal{P}_3'')) = 4/3$.

Conjecture 5 $\overline{\rho}(A(\mathcal{P}_4); A(\mathcal{P}_4'')) = 2/\sqrt{3}$.

This conjecture is motivated by the polynomial $p(z) = (z^2 + \frac{2}{3}z + 1)(z-1)^2$, for which $\overline{\rho}(A(p); A(p'')) = 2/\sqrt{3}$.

3.3 Deviation of $A(p)$ from $A(p^{(n-1)})$ and $A(p^{(n-2)})$ for a polynomial $p \in \mathcal{P}_n$

It is trivial that

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(n-1)})) = 2 - \frac{2}{n}.$$

An extremal polynomial is $p(z) = (z-1)(z+1)^{n-1}$.

Conjecture 6 *For every $p \in \mathcal{P}_n$ and $1 \leq s \leq n-1$, the inequality*

$$\rho(A(p); A(p^{(s)})) \leq \frac{2s}{s+1}$$

holds.

Lemma 3.1 *For $n \geq 5$, the inequalities*

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(n-2)})) \geq \begin{cases} 2\frac{n-1}{n+1} & \text{for } n \text{ odd,} \\ 1 + \sqrt{\frac{(n-2)(n-4}{n(n+2)}} & \text{for } n \text{ even} \end{cases}$$

hold.

Proof. For $n = 2m+1$, consider the polynomial

$$p(z) = \left(z^2 + \frac{2m}{m+1}z + 1\right)^m (z-1)$$

with

$$p^{(2m-1)}(z) = \frac{(2m+1)!}{2}\left(z + \frac{m-1}{m+1}\right)^2.$$

$p^{(2m-1)}$ has a double zero $\zeta_1 = \zeta_2 = -(m-1)/(m+1)$ and

$$\rho(A(p); A(p^{(n-2)})) = |1-\zeta_1| = \frac{2m}{m+1} = 2\frac{n-1}{n+1}.$$

For $n = 2m+2$, consider the polynomial

$$p(z) = \left(z^2 + 2\sqrt{\frac{m^2-1}{m(m+2)}}z + 1\right)^m (z^2-1)$$

with

$$p^{(2m)}(z) = \frac{(2m+2)!}{2}\left(z + \sqrt{\frac{m(m-1)}{(m+1)(m+2)}}\right)^2.$$

$p^{(2m)}$ has a double zero $\zeta_1 = \zeta_2 = -\sqrt{m(m-1)/(m+1)(m+2)}$ and

$$\rho(A(p); A(p^{(n-2)})) = |1-\zeta_1| = 1 = \sqrt{\frac{(n-2)(n-4)}{n(n+2)}}. \qquad \square$$

3.4 Deviation of $A(p)$ from $H(p')$ and $D_c(p')$

From (3.5) and Lemma 2.1 there follows that

$$\rho(A(p); A(p')) \geq \rho(A(p); H(p')) \geq \rho(A(p); D_c(p')). \tag{3.8}$$

Then we formulate a weaker conjecture than Conjecture 1.

Conjecture 7 *For every $n \geq 2$, the following equalities hold:*

$$\overline{\rho}(A(\mathcal{P}_n); D_c(\mathcal{P}'_n)) = 1 \quad \text{and} \quad \overline{\rho}(A(\mathcal{P}_n); H(\mathcal{P}'_n)) = 1. \tag{3.9}$$

The equalities (3.9) are motivated by (3.8) and by the polynomial $p(z) = z^n - 1 \in \mathcal{P}_n$.

3.5 Deviations of $H(p)$

In 1977, G. Schmeisser formulated

Conjecture 8 (Schmeisser's Conjecture [38]). *For every $n \geq 2$, the equality $\overline{\rho}(H(\mathcal{P}_n); A(\mathcal{P}'_n)) = 1$ holds.*

We conjecture also that

$$\overline{\rho}(H(\mathcal{P}_n); D_c(\mathcal{P}'_n)) = 1 \quad \text{and} \quad \overline{\rho}(H(\mathcal{P}_n); H(\mathcal{P}'_n)) = 1$$

for every $n \geq 2$.

3.6 Deviations of D

It is obvious that $\overline{\rho}(D; A(\mathcal{P}'_2)) = 1$.

Conjecture 9 *For every $n \geq 3$, the equality*

$$\overline{\rho}(D; A(\mathcal{P}'_n)) = \sqrt{2} \tag{3.10}$$

holds.

A motivation for Conjecture 9 is the polynomial

$$p(z) = (z^2 + 1)(z - 1)^{n-2} \in \mathcal{P}_n. \tag{3.11}$$

The zeros of the derivative $p'(z) = (nz^2 - 2z + n - 2)(z + 1)^{n-3}$ are

$$\zeta_1 = (1 + i\sqrt{n^2 - 2n - 1})/n, \quad \zeta_2 = (1 - i\sqrt{n^2 - 2n - 1})/n \tag{3.12}$$

and $\zeta_j = 1$ for $j = 3, 4, \ldots, n-1$. A simple calculation gives $\rho(D; A(p')) \geq |1 + \zeta_1| = \sqrt{2}$, hence

$$\overline{\rho}(D; A(\mathcal{P}'_n)) \geq \sqrt{2}. \tag{3.13}$$

Theorem 3.7 *Conjectute 9 is true for $n = 3$.*

Proof. Let $\overline{p} \in \mathcal{P}_3$ be an extremal polynomial for $\overline{\rho}(D; A(\mathcal{P}'_3))$. By Corollary 2.2 we may suppose that $\overline{p}(z) = (z^2 - 2\alpha z + 1)(z - z_3)$, where $\alpha \geq 0$. Let $\overline{p}'(z) = 3(z - \zeta_1)(z - \zeta_2)$ and suppose that $\rho(1; H(\overline{p}')) > \sqrt{2}$. Then $|1 - \zeta_1| > \sqrt{2}$, $|1 - \zeta_2| > \sqrt{2}$ and

$$|\overline{p}(1)| = 3|1 - \zeta_1||1 - \zeta_2| > 6.$$

On the other hand

$$|\overline{p}'(1)| = |3 - 2(2\alpha + z_3) + 1 + 2\alpha z_3| = 2|1 - \alpha||2 - z_3| \leq 6. \qquad \square$$

The Hausdorff deviation $\rho(D; H(p'))$ is larger than 1 for some polynomials $p \in \mathcal{P}$.

Conjecture 10 *For every $n \geq 3$, the equality*

$$\overline{\rho}(D; H(\mathcal{P}'_n)) = 1 + \frac{1}{n} \tag{3.14}$$

holds.

A motivation for Conjecture 10 is also the polynomial (3.11), from which we have that

$$\overline{\rho}(D; H(\mathcal{P}'_n)) \geq 1 + \frac{1}{n}. \tag{3.15}$$

Theorem 3.8 *Conjectute 10 is true for $n = 3$.*

Proof. As in the proof of Theorem 3.7, let $\overline{p} \in \mathcal{P}_3$ be an extremal polynomial for $\overline{\rho}(D; H(\mathcal{P}_3'))$ and $\overline{p}(z) = (z^2 - 2\alpha z + 1)(z - z_3)$, where $\alpha \geq 0$. Let $\overline{p}'(z) = 3(z - \zeta_1)(z - \zeta_2)$ and suppose that $\rho(1; H(\overline{p}')) > 4/3$. Then

$$\left|1 - \frac{\zeta_1 + \zeta_2}{2}\right| > \frac{4}{3}.$$

On the other hand

$$\overline{p}'(z) = 3z^2 - 2(2\alpha + z_3)z + 1 + 2\alpha z_3,$$

hence $\zeta_1 + \zeta_2 = \frac{2}{3}(2\alpha + z_3)$ and

$$\left|1 - \frac{1}{3}(2\alpha + z_3)\right| = \frac{1}{3}|2(1 - \alpha) + (1 - z_3)| \leq \frac{4}{3}. \qquad \square$$

For the Hausdorff deviation $\rho(D; D_c(p'))$ we formulate

Conjecture 11 *For every $n \geq 3$, the equality $\overline{\rho}(D; D_c(\mathcal{P}_n')) = 1$ holds.*

Theorem 3.9 *Conjectute 11 is true for $n = 3$.*

Proof. As in the proof of Theorem 3.7 and Theorem 3.8, let $\overline{p} \in \mathcal{P}_3$ be an extremal polynomial for $\overline{\rho}(D; D(\mathcal{P}_3'))$ and $\overline{p}(z) = (z^2 - 2\alpha z + 1)(z - z_3)$, where $\alpha \geq 0$. Let $\overline{p}'(z) = 3(z - \zeta_1)(z - \zeta_2)$ and suppose that $\rho(1; D(\overline{p}')) > 1$. Then

$$\left|1 - \frac{\zeta_1 + \zeta_2}{2}\right| - \left|\frac{\zeta_1 + \zeta_2}{2}\right| > 1$$

and

$$\left|1 - \frac{1}{3}(2\alpha + z_3)\right| - \frac{1}{3}|2\alpha + z_3)| > 1.$$

On the other hand

$$\left|1 - \frac{1}{3}(2\alpha + z_3)\right| \leq 1 + \frac{1}{3}|2\alpha + z_3)|,$$

which is contradiction. $\square$

4 Weaker estimates for $\rho(A(p); A(p'))$

In 1972, G. Schmeisser [37, p. 406, Problem 2] formulated

Problem 1 (G. Schmeisser [37]). *Find a constant ρ (as small as possible), such that for every $p \in \mathcal{P}_n$, the inequality*

$$\rho(A(p); A(p')) \leq \rho \tag{4.16}$$

holds.

From Gauss–Lucas Theorem 2.1 there follows that $\rho \leq 2$. In G. Schmeisser (1971) [37, p. 413, Satz 2] it is proved that $\rho \leq 1.568$.

In 1982 M. Marden [24, p. 124, Theorem 1] proved

Theorem 4.1 (M. Marden [24]). *If $p \in \mathcal{P}_n$ and $M = \sup\{|p(z)| : |z| = 1\}$, then the inequality*

$$\rho(A(p); A(p')) \leq M^{1/(n-1)} \tag{4.17}$$

holds.

M. Marden [24, p. 126] claims that Theorem 4.1 validates Conjecture 1 "*in the general case when $n \to \infty$*".

The best estimate for ρ till now is given in Bojanov, B. B., Q. I. Rahman and J. Szynal (1985) [4], and it is based on the following

Theorem 4.2 (B. D. Bojanov, Q. I. Rahman and J. Szynal [4]). *For every $p \in \mathcal{P}_n$, the inequality*

$$\rho(A(p); A(p')) \leq \rho_n = (1 + |z_1 z_2 \cdots z_n|)^{1/n}$$

holds.

After J. E. Brown and G. Xiang [11] proved Conjecture 1 for $n \leq 8$, from Theorem 4.2 there follows

Corollary 4.1 (B. D. Bojanov, Q. I. Rahman and J. Szynal). *For every $p \in \mathcal{P}$, the inequality*

$$\rho(A(p); A(p')) \leq 1.08006\ldots$$

holds.

We have also

Corollary 4.2 (Bojanov, B. D., Q. I. Rahman and J. Szynal). *For every $p \in \mathcal{P}$, the equality*

$$\lim_{n\to\infty} \rho(A(p); A(p')) = 1$$

holds.

In other words, Conjecture 1 is asymptotically true.

References

[1] Alexander, J. W.: *Functions, which map the interior of the unit circle upon simple region*, Ann. of Math., **17** (1915), 12 - 22.

[2] Aziz, A.: *On the zeros of a polynomial and its derivative*, Bull. Austr. Math. Soc. **31**, # 4 (1985), 245 - 255.

[3] Biernacki, M: *Sur le functions à seriés lacunaire*, C. r. Acad. Sci. Paris Ser. A-B **187** (1928), 477 - 479.

[4] Bojanov, B. D., Q. I. Rahman and J. Szynal: *On a conjecture of Sendov about the critical points of a polynomial*, Math. Z. **190** (1985), 281 - 285.

[5] Borcea, J.: *On the Sendov conjecture for polynomials with at most six distinct zeros*, J. Math. Anal. Appl. **200**, #1 (1996), 182 - 206.

[6] Borcea, J.: *The Sendov conjecture for polynomials with at most seven distinct roots*, Analysis **16** (1996), 137 - 159.

[7] Brannan, D. A.: *On a conjecture of Ilieff*, Math. Proc. Cambridge Phil. Soc. **64** (1968), 83 - 85.

[8] Brown, J. E.: *On the Sendov conjecture for sixth degree polynomials*, Proc. Amer. Math. Soc. **113**, # 4 (1991), 939 - 946.

[9] Brown, J. E.: *A proof of the Sendov conjecture for polynomials of degree seven.* Complex Variables Theory Appl. **33**, # 1-4 (1997), 75–95.

[10] Brown, J. E.: *On the Sendov's Conjecture for polynomials with real critical points*, Contemporary Mathematics **252** (1999), 49 - 62.

[11] Brown, J. E. and G. Xiang: *Proof of the Sendov conjecture for polynomials of degree at most eight*, J. Math. Anal. Appl. **232**, # 2 (1999), 272–292.

[12] Byrne, A.: *Some results for Sendov conjecture*, J. Math Anal. Appl. **199** (1996), 754 - 768.

[13] Cohen, G. L. and G. H. Smith: *A proof of Iliev's conjecture for polynomials with four zeros*, Elemente d. Math. **43** (1988), 18 - 21.

[14] Cohen, G. L. and G. H. Smith: *A simple verification of Iliev's conjecture for polynomials with three zeros*, Amer. Math. Mountly **95** (1988), 734 - 737.

[15] Dieudonné, J.: *Sur quuelques applications de la théorie des functions borneés aux polynómes dont toutes les racines sont dans un domain circulaire donné*, Actualités Scient. et Industri **144**, (1934), 5 - 24.

[16] Dronka J. and J. Sokól: *On the location of critical points of some complex polynomials*. Demonstratio Math. **32**, # 2 (1999), 297–302.

[17] Gacs, G.: *On polynomials whose zeros are in the unit disk*, J. Math. Anal. Appl. **36** (1971), 627 - 637.

[18] Goodman, A. G., Q. I. Rahman and J. Ratti: *On the zeros of a polynomial and its derivative*, Proc. Amer. Math. Soc. **21** (1969), 273 - 274.

[19] Hayman, W. K.: *Research Problems in Function Theory*, Althlone Press, London, 1967.

[20] Joyal, A.: *On the zeros of a polynomial and its derivative*, J. Math. Anal. Appl. **25** (1969), 315 - 317.

[21] Kakeya, S.: *On zeros of polynomial and its derivative*, Tôhoku Math. J. **11** (1917), 5 - 16.

[22] Kumar, S. and B. G. Shenoy: *On the Sendov–Ilieff conjecture for polynomials with at most five zeros*, J. Math. Anal. Appl. **171** (1992), 595 - 600.

[23] Marden, M.: *Geometry of Polynomials*, 2nd edition, Math. Surveys Monographs, **3**, Amer. Math. Soc., Providence, RI, 1966.

[24] Marden, M.: *On the critical points of a polynomial*, Tensor (N. S.) **39** (1982), 124 - 126.

[25] Marden, M.: *The search for a Roll's theorem in the complex plain*, Amer. Math. Monthly **92** (1984), 643 - 650.

[26] McCoy, T. L.: *A principal of O. Szász and the Sendov–Ilyeff problem for polynomials near* $z^n - 1$, Complex Variables Theory Appl. **35**, # 4 (1998), 121–155.

[27] Meir, A. and A. Sharma: *On Ilieff's conjecture*, Pacific J. Math. **31** (1969), 459 - 467.

[28] Miller, M. J.: *On Sendov's conjecture for roots near the unit circle*, J. Math. Anal. Appl. **175** (1993), 632 - 639.

[29] Pflug, P. and G. Schmieder: *Remarks on Ilieff–Sendov Problem*, Annals Univ. Mariae Curie-Sklodowska Lublin, Poland, **XLVIII** # 9, (1994), 98 - 105.

[30] Phelps, D. and R. S. Rodriguez: *Some properties of extremal polynomials for the Ilieff conjecture*, Kodai Math. Sem. Rep. **24** (1972), 172 - 175.

[31] Rahman, Q. I.: *On the zeroes of a polynomial and its derivative*, Pacific J. Math. **41** (1972), 525 - 528.

[32] Rahman, Q. I. and Q. M. Tariq: *On a problem related to the conjecture of Sendov about the critical points of a polynomial*, Canad. Math. Bull, **30**, # 4 (1987), 476 - 480.

[33] Rassias, Th. M.: *Zeros of polynomials and their derivative*, Rev. Roumaine Math. Purres Appl. **36** (1991), 441 - 448.

[34] Rubinstein, Z.: *On a problem of Ilieff*, Pacific J. Math. **26** (1968), 159 - 161.

[35] Saff, E. B. and J. B. Twomey: *A note on the location of critical points of polynomials*, Proc. Amer. Math. Soc. **27**, # 2 (1971), 303 - 308.

[36] Schmeisser, G.: *Bemerkungen zu einer Vermutung von Ilieff*, Math. Z. **111** (1969), 121 - 125.

[37] Schmeisser, G: *Zur Lage der Kritichen puncte eines polynoms*, Rend. Sem. Mat. Univ. Padova **46** (1971), 165 -173.

[38] Schmeisser, G: *On Ilieff's conjecture*, Math. Z. **156** (1977), 165 - 173.

[39] Sendov, Bl.: *Hausdorff Geometry of Polynomials*, East Journal on Approximation, **7** # 2 (2001), 1 - 56.

[40] Szegó, G.: *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math Z. **13** (1922), 28 - 55.

[41] Tariq, Q. M.: *On the zeros of a polynomial and its derivative I*, J. Univ. Kuwait (Sci.) **13**, (1986), 17 - 19.

[42] Tariq, Q. M.: *On the zeros of a polynomial and its derivative II*, J. Univ. Kuwait (Sci.) **13**, (1986), 151 - 155.

[43] Todorov, P. G.: *A natural verification of the Sendov conjecture for the canonical cubic equation and other results for the location of its roots*, Bull. de la Classe des Sciences, **VII** (1996), 387 - 403.

[44] Vernon, S.: *On the critical points of polynomials*, Proc. Roy. Irish Acad. Sect. A, **78** (1978), 195 - 198.

[45] Vâjâtu, V. and A. Zaharescu: *Ilyeff's conjecture on a corona.* Bull. London Math. Soc. **25**, # 1 (1993), 49 - 54.

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