Hausdorff Geometry of Polynomials

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Abstract

In this paper, we give a review of problems from the Geometry of polynomials that are related to the following

Conjecture 1 *If all the zeros of the polynomial* $p(z) := \prod_{k=1}^{n}(z - z_k), \quad (n \geq 2)$ *lie in the unit disk* $D(0,1) = \{z : \ |z| \leq 1\}$, *then for every* z_k *the disk* $D(z_k, 1)$ *contains at least one zero of* $p'(z)$.

Among various results, we distinguish Rolle-type statements, which involve only two zeros of a polynomial and one critical point of this polynomial, and so called Hausdorff-type statements, which involve the set of all zeros and the set of all critical points of a polynomial. The first example of a Hausdorff-type theorem is the famous Gauss–Lucas Theorem.

Conjecture 1 is formulated as an estimate of the Hausdorff deviation of the set of zeros from the set of critical points of a polynomial. We review all results, conjectures and problems related to Conjecture 1, known to the author, that have been published in the last century. Some new conjectures in Hausdorff geometry of polynomials are formulated. All found publications, related to Conjecture 1, are listed separately in the references and all found partial proofs of this conjecture are listed in the Appendices.

Conjecture 1 entered the literature by mistake as "Ilieff's Conjecture". The autobiographical note, given in the last Appendix, explains how it has been formulated and why it has been attributed to L. Ilieff (1913 - 2000).

Key words: *Geometry of polynomials, Hausdorff geometry of polynomials, zeros, critical points, Rolle-type theorems.*

Mathematics Subject Classification: 30C10

1 Introduction

In 1958, as an assistant of Professor N. Obreshkov, I formulated the folloing problem in the Geometry of Polynomials.

Conjecture 1 *If all the zeros of the polynomial*

$$p(z) := \prod_{k=1}^{n} (z - z_k), \quad (n \geq 2)$$

lie in the unit disk $D = D(0,1) = \{z : |z| \leq 1\}$*, then for every* z_k *the disk* $D(z_k, 1) = \{z : |z - z_k| \leq 1\}$ *contains at least one zero of* $p'(z)$*.*

The shortest explanation of the origin of Conjecture 1 is given by M. Marden [56, p. 267]: "*This conjecture was included in the collection of Research Problems in Function Theory, published in 1967 by Professor Hayman* [40]*. Since it had been brought to Hayman's attention by Professor Ilyeff, it became known as "Ilyeff's conjecture. Actually, Conjecture 1 was due to the Bulgarian mathematician Bl. Sendov who had acquainted me and probably others with it in 1962 at the International Congress of Mathematics held in Stockholm.*" (See **10.2**).

Up to the end of the XXth century, Conjecture 1 was neither proved nor disproved, although more than 70 related papers have been published on this topic. In attempts to prove this conjecture, many new results have been obtained and many other problems and conjectures have been formulated. Our purpose in this paper is to review these results, problems and conjectures from the point of view of Hausdorff distance between the set of zeros and the set of critical points of a polynomial.

To every bounded set of points A in the complex plane $\mathcal{C}$ one can associate in a unique way other sets such as the *convex hull* $H(A)$ of A, the *circumself disk* $D_c(A)$ of A and the *gravity disk* $D_g(A)$ of A (see **2.2**), which give information for the location of A . Using these sets we formulate new theorems, problems and conjectures in terms of the Hausdorff deviation and the Hausdorff distance [85] (see **2.1**). Here we have to mention the comment of G. Schmieder [82, p. 497]: "*We observe two different heuristic principles for the distribution of the derivative zeros. Zeros of* p' *prefer a neighborhood of a cluster of zeros of* p*, as well as a lone zero of* p*.*" These two heuristic principles of G. Schmieder are quantified by the Hausdorff distance between the set of zeros and the set of critical points of a polynomial.

An interesting conjecture, stronger than Conjecture 1, announced in A. G. Goodman, Q. I. Rahman and J. Ratti (1969) [39] and independently in G. Schmeisser (1969) [77], is the following [1]

[1]In G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias (1994) [64, p. 222], the following footnote appears: "As A. Goodman informed one of the authors (in a letter dated June 20, 1992), the conjecture is due only to J. S. Ratti, who wanted to include it in their joint paper... Thus the *Goodman-Rachman-Ratti conjecture* should be called the *Ratti-Schmeisser conjecture."*

Conjecture 2 (Ratti-Schmeisser Conjecture). *If all the zeros of the polynomial*

$$p(z) := \prod_{k=1}^{n}(z - z_k), \quad (n \geq 2)$$

lie in the unit disk $D(0,1)$*, then for every* z_k *the disk* $D(z_k/2, 1 - |z_k|/2)$ *contains at least one zero of* $p'(z)$*.*

Obviously, Conjecture 1 follows from Conjecture 2, as the disk $D(z_k/2, 1-|z_k|/2)$ lies in the disk $D(z_k, 1)$. Conjecture 2 was proved by G. Gacs (1971) [36] for $2 \leq n \leq 5$. M. J. Miller (1990) [60], using computers, constructed counterexamples for Conjecture 2 for polynomials of degree 6, 8, 10 and 12. Counterexamples for polynomials of degree 7, 9 and 11 were found by S. Kumar and B. G. Shenoy (1991) [49].

Conjecture 2 is true for a zero z_k with $|z_k| = 1$, as shown in [39], [77], see Lemma 5.1.

A natural inverse problem of Conjecture 1 was formulated and solved by A. Aziz (1985) [13].

Theorem 1.1 (Aziz' Theorem). *If all the zeros of the polynomial* $p(z)$ *of degree* $n \geq 2$ *lie in the unit disk* D *and* ζ_j *is a zero of* $p'(z)$*, then* $p(z)$ *has at least one zero in the disk* $D_1 = \{z : |z - \zeta_j| \leq 1\}$ *and even in the disk* $D_2 = \{z : |z - 2\zeta_j| \leq 1\}$*.*

Observe that $(D_2 \bigcap D) \subset (D_1 \bigcap D)$. A simple geometric proof of Theorem 1.1 follows from the Gauss-Lucas Theorem, see Theorem 3.3.

Conjecture 1 estimates the *Hausdorff deviation* (see **2.1**) of the set of the zeros from the set of the critical points. Aziz' Theorem 1.1 estimates the *Hausdorff deviation* of the set of the critical points from the set of the zeros. This observation motivates the idea to consider the Hausdorff distance between these two sets. This idea is mentioned by B. E. Petersen in a Worksheet (2000) [65].

In Section 2 we define the Hausdorff deviation from one set of points to another set of points in the complex plane. By this deviation the Hausdorff distance between two point sets is defined. Our aim is to study the geometric interrelations between the set of the zeros and the set of the critical points of a polynomial. For this purpose we use three neighborhoods of a point set A: the convex hull $H(A)$, the circumself disk $D_c(A)$ and the gravity disk $D_g(A)$, see **2.2**. These neighborhoods give information for the location of the set A. Conjecture 1 is formulated as an estimate of the Hausdorff deviation of the set of the zeros of a polynomial from the set of the critical points of this polynomial. Several classes of polynomials are defined to facilitate the formulation of different statements. The basic theorems from the Geometry of Polynomials, used in the Hausdorff geometry of polynomials, are formulated without proofs.

In Section 3 we consider the deviation of a given neighborhood of the zeros of a polynomial from a given neighborhood of the critical points of this polynomial and vice versa. Let $A(p)$ be the set of all distinct zeros of a polynomial p and $A(p')$ be the set of all distinct zeros of its derivative. We consider all possible cases of a deviation between one of the sets $A(p), H(p) = H(A(p)), D_c(p) = D_c(A(p)), D_g(p) = D_g(A(p))$ and one of the sets $A(p'), H(p'), D_c(p'), D_g(p')$. Conjecture 1 concerns the Hausdorff deviation of the set $A(p)$ from the set $A(p')$. The Aziz' Theorem 1.1 gives an exact estimate for the Hausdorff deviation of the set $A(p')$ from the set $A(p)$. Using the neighborhoods defined above, we formulate several conjectures, which follow from Conjecture 1 (weaker than Conjecture 1). In our opinion, they are far from trivial, see for example conjectures 3, 5, and 13. We also give conjectures, which are stronger than Conjecture 1, see conjectures 7, 9 and the new conjectures such as Conjecture 10.

The extremal polynomials for a given Hausdorff deviation are considered in Section 4. Let $F(p)$ be a continuous functional defined in a compact set $\mathcal{U}$ of polynomials p, and let $\overline{F}(\mathcal{U}) = \sup\{F(p) : \ p \in \mathcal{U}\}$ and $\underline{F}(\mathcal{U}) = \inf\{F(p) : \ p \in \mathcal{U}\}$. A polynomial $p \in \mathcal{U}$ with $F(p) = \overline{F}(\mathcal{U})$ is called supremal in $\mathcal{U}$ for F and a polynomial $p \in \mathcal{U}$ with $F(p) = \underline{F}(\mathcal{U})$ is called infiemal in $\mathcal{U}$ for F. We consider extremal polynomials (supremal and infimal) in different compact sets of polynomials for different Hausdorff deviations, which are continuous functionals. Several results for the properties of the extremal polynomials are formulated.

We call a zero of a polynomial *normal*, if Conjecture 1 is true for this zero. In Section 5 we review results that are related to a normal zero for a polynomial. A zero on the unit disk and a zero in the origin is a normal zero. If a zero is sufficiently close to the unit disk, it is also a normal zero.

Section 6 is devoted to the classes of polynomials for which Conjecture 1 is true. The first results are from 1968, after the publication in 1967 of the Hayman's book [40]. In a period of one year, the Conjecture 1 was proved for polynomials of degree 3, 4, 5 and for polynomials of an arbitrary degree with not more then 3, 4, or 5 distinct zeros. After a break of more than 20 years, it was proved for polynomials of 6 and 7 degree. The final result in this saga belongs to J. E. Brown and G. Xiang (1999) [27], who proved Conjecture 1 for all polynomials with not more than 8 distinct zeros. Conjecture 1 is proved also for some lacunary polynomials and polynomials with special shape of the convex hull $H(p)$ of their zeros.

Section 7 is devoted to the problem of G. Schmeisser (1972) [78, p. 406, Problem 2] to find a constant ρ (as small as possible), such that if all the zeros $\{z_k\}$ of a polynomial p lie in the unit disk D, then for every zero z_k of p, there exists at least one critical point of p, which lies in the disk $D(z_k, \rho)$. From Gauss–Lucas Theorem 2.1 $\rho \leq 2$ follows. In G. Schmeisser (1971) [78, p. 413, Satz 2] it is proved that $\rho \leq 1.568$. The best estimate for ρ, till now, is given in B. B. Bojanov, Q. I. Rahman and J. Szynal (1985) [15], see Theorem 7.2, from which it follows that $\rho \leq 1.08006\ldots$.

In Section 8 we consider the Hausdorff geometry of special classes of polynomials: polynomials with real critical points, polynomials with real zeros and polynomials with zeros in a half-disk.

Untill Section 9 we consider only the geometric relations between the zeros of a polynomial and the zeros of its first derivative. In Section 9 we turn to the zeros of all derivatives of the considered polynomial. This is a new direction with very little results. In this section we also review other problems connected with Conjecture 1.

In the Appendices we give a full list of the results by dates, from the accessible publications, connected with Conjecture 1. At the end, an autobiographical note is given for two reasons: first, to explain how I have come to Conjecture 1, and second, as a justification, why I have done almost nothing to try and prove this conjecture.

2 Preliminaries and notations

In this section we define the Hausdorff distance between two bounded and closed sets of points in the complex plane. This distance is defined by the Hausdorff deviation of one set from another. Then we adopt notations for sets of algebraic polynomials.

2.1 Hausdorff distance in the complex plane

Let (M, ρ) be a metric space, where $\rho(a, b)$ is the distance between two elements $a, b \in M$. Usually the distance between the point $b \in M$ and the set $A \subset M$ is defined as $\rho(b; A) = \inf_{a \in A} \rho(a, b)$. One may define a "distance" between two subsets $A, B \subset M$ as

$$d(A, B) = \inf_{a \in A} \rho(a; B) = \inf_{b \in B} \rho(b; A). \tag{2.1}$$

The "distance" (2.1) is not a true distance. It is not difficult to see that $\rho(A, B) = 0$ when A and B have only a common point or a common limit point.

In his famous book, F. Hausdorff [6] defined a distance between subsets of a metric space in the following way. Let $A \subset M$. For $\alpha \geq 0$ denote by $U(\alpha, A)$ the set of all points $x \in M$ such that $\rho(x; A) \leq \alpha$, i. e.,

$$U(\alpha, A) = \{x : x \in M \text{ and } \rho(x; A) \leq \alpha\}.$$

Definition 2.1 *The infimum of those α for which $U(\alpha, A) \supset B$ and $U(\alpha, B) \supset A$ is called the* **Hausdorff distance** *or* **distance in the sense of deviation** *between the sets A and B, induced by the distance ρ.*

If M is a complete space and 2^M is the set of all compact subsets of M, then the Hausdorff distance between subsets is a metric in 2^M.[2]

We shall use the Hausdorff distance in the space $2^{\mathcal{C}}$ of all bounded and closed point sets on the complex plane $\mathcal{C}$.

Definition 2.2 *For $A, B \in 2^{\mathcal{C}}$ set:*

[2] It is possible to define Hausdorff metric in functional spaces, see for example [10], [11].

1. $\rho(b; A) = \inf\{|b-a| : \ a \in A\}$,
2. $\rho(B; A) = \sup\{\rho(b; A) : \ b \in B\}$ - *the* **Hausdorff deviation** *of B from A,*
3. $h(A, B) = \max\{\rho(A; B), \rho(B; A)\}$ - *the* **Hausdorff distance** *between A and B.*

Obviously, Definition 2.2 and Definition 2.1 are equivalent. From the properties of the Hausdorff distance in the set of all compact subsets of a complete space there follows that

$h(A, B) = h(B, A), \quad h(A, C) \leq h(A, B) + h(B, C)$ and $h(A, B) = 0 \Rightarrow A = B$.

A simple and useful statement is

Lemma 2.1 *If $A, B, V \in 2^{\mathcal{C}}$ and $B \subset V$, then*

$$\rho(A; B) \geq \rho(A; V) \quad \text{and} \quad \rho(B; A) \leq \rho(V; A).$$

Proof. From the definition

$$\rho(A; B) = \sup\{\rho(a; B) : \ a \in A\} \geq \sup\{\rho(a; V) : \ a \in A\} = \rho(A; V),$$

as $B \subset V$ and

$$\rho(a; B) = \inf\{|a-b| : \ b \in B\} \geq \inf\{|a-v| : \ v \in V\} = \rho(a; V).$$

In the same way

$$\rho(V; A) = \sup\{\rho(v; A) : \ v \in V\} \geq \sup\{\rho(b; A) : \ b \in B\} = \rho(B; A),$$

as $B \subset V$. $\square$

In general $\rho(B; A) \neq \rho(A; B)$.

2.2 Three closed and convex neighborhoods of a set of points

Let $\mathcal{A}_m$ be the set of all sets $A = \{a_1, a_2, \ldots, a_m\}$ of m distinct points in $\mathcal{C}$ and $\mathcal{A} = \bigcup_{n=2}^{\infty} \mathcal{A}_n$. To locate a set $A \in \mathcal{A}$ in the complex plane, we use three uniquely defined closed and convex sets in $\mathcal{C}$, which contain A.

2.2.1 Convex hull

Definition 2.3 *For every $A \in \mathcal{A}$, the smallest closed and convex point set $H(A) \subset \mathcal{C}$, which contains A is called the* **convex hull** *of A. If $A \in \mathcal{A}_n$, then $H(A)$ is a convex m-angle with $m \leq n$. Call the apices of $H(A)$,* **corners** *of A and denote by $\hat{A} \subset A$ the set of all corners of A.*

Lemma 2.2 *Let A and B be two bounded and closed point sets in the complex plane. Then*

$$h(H(A), H(B)) \leq h(A, B). \tag{2.2}$$

Proof. It is sufficient to prove (2.2) when $A = \{a_1, a_2\}; \quad a_1 \neq a_2$ and $B = \{b_1, b_2\}; \quad b_1 \neq b_2$, which is obvious. $\square$

2.2.2 Circumself disk

Definition 2.4 *For every $A \in \mathcal{A}$ define:*

1. $D_c(A) = D(c_c(A), r_c(A))$ *- the* **circumself disk** *of A, i. e., the smallest closed disk, which contains A;*
2. $c_c(A)$ *- the* **circumself center** *of A;*
3. $r_c(A)$ *- the* **circumself radius** *of A;*
4. $D = D(0,1)$ *- the* **unit disk***;*
5. $C_c(A) = \{z : |z - c_c(A)| = r_c(A))$ *- the* **circumself circle**[3] *of A;*
6. $C = \{z : |z| = 1\}$ *- the* **unit circle***.*

The following relations are obvious for every $A \in \mathcal{A}$:

$$A \subset H(A) \subset D_c(A). \tag{2.3}$$

From the definition of circumself disk of a set $A \in \mathcal{A}$ there follows

Lemma 2.3 *If $A \in \mathcal{A}$, then:*

1. *At least two distinct points of A lie in $C_c(A)$;*
2. *At least one point of A lies in each subarc of $C_c(A)$ with length π.*

Lemma 2.4 *Let $A \subset D$, $b \notin H(A)$ and $\rho(b; H(A)) \leq 1$. Then $\rho(b; A) \leq \sqrt{2}$.*

Proof. Let a be the unique point for which $|a - b| = \rho(b; H(A))$. Observe that $H(A)$ is convex. Let l be the support line of $H(A)$, which separates b from $H(A)$ and, which is orthogonal to the line segment $[a, b]$. If $a \in A$, then $\rho(b; A) = \rho(b; H(A)) \leq 1$. If $a \notin A$, then there are at least two points $a_1, a_2 \in A$, which lie on l and a lies in the segment $[a_1, a_2]$. Let $|a - a_1| \leq |a - a_2|$, then $|a - a_1| \leq 1$, as $|a_1 - a_2| \leq 2$. Then $|b - a_1| = \sqrt{|b - a|^2 + |a - a_1|^2} \leq \sqrt{2}$. □

[3]Note the difference between circumcircle of a triangle and circumself circle

2.2.3 Gravity disk

Definition 2.5 *For every* $A = \{a_1, a_2, \dots, a_n\} \in \mathcal{A}$ *define:*

1. $c_g(A) = (a_1 + a_2 + \cdots + a_n)/n$ - *the* **gravity center** *of* A*;*
2. $D_g(A) = D(c_g(A), r_g(A))$ - *the* **gravity disk** *of* A*, i. e., the smallest closed disk, with center* $c_g(A)$*, which contains* A*;*
3. $r_g(A)$ - *the* **gravity radius** *of* A*;*
4. $C_g(A) = \{z : |z - c_g(A)| = r_c(A))$ - *the* **gravity circle** *of* A*.*

The following relations are obvious for every $A \in \mathcal{A}$

$$A \subset H(A) \subset D_g(A). \tag{2.4}$$

From the definition of gravity disk of a set $A \in \mathcal{A}$ there follows:

Lemma 2.5 *If* $A \in \mathcal{A}$*, then at least one point of* A *lies in* $C_g(A)$ *and* $r_c(A) \le r_g(A) < 2r_c(A)$*.*

2.3 Polynomial sets

As we are interested only in the relative location of the zeros and the critical points of a polynomial, it is sufficient to consider monic polynomials $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$.

Definition 2.6 *Set the following notations:*

1. *Let* $\mathcal{U}$ *be a set of polynomials, then denote by* $\mathcal{U}(a)$ *the set of all polynomials* $p \in \mathcal{U}$ *with* $p(a) = 0$*;*
2. $\mathcal{U}^{(s)}$ - *the set of the* s^{th} *derivatives of the polynomials* $p \in \mathcal{U}$ $(\mathcal{U}^{(1)} = \mathcal{U}')$*;*
3. $\mathcal{M}$ - *the set of all monic polynomials of degree* $n \ge 2$*;*
4. $\mathcal{M}_n$ - *the set of all polynomials* $p \in \mathcal{M}$ *of degree* n*;*
5. $\mathcal{M}_{n,m}$*;* $m \le n$ - *the set of all polynomials* $p \in \mathcal{M}_n$ *of the form* $p(z) = \prod_{k=1}^{m}(z - z_k)^{n_k}$*, where* $n = n_1 + n_2 + \cdots + n_m$ *and* $z_1, z_2, \dots, z_m$ *are distinct;*
6. $A(p) = \{z_1, z_2, \dots, z_m\}$ - *the set of all distinct zeros of* $p \in \mathcal{M}_{n,m}$*;*
7. $A_q(p)$ - *the set of all distinct zeros of* p*, which are not zeros of* q*;*
8. $H(p) = H(A(p))$ - *the* **convex hull** *of* p*;*

9. $D_c(p) = D_c(A(p))$ - *the* **circumself disk** *of* p;

10. $c_c(p) = c_c(A(p))$ - *the* **circumself center** *of* p;

11. $r_c(p) = r_c(A(p))$ - *the* **circumself radius** *of* p;

12. $c_g(p) = (n_1 z_1 + n_2 z_2 + \cdots + n_m z_m)/n$ - *the* **gravity center** *of* $p \in \mathcal{M}_{n,m}$;

13. $D_g(p) = D(c_g(p), r_g(p))$ - *the* **gravity disk** *of* p, *i. e., the smallest closed disk with center* $c_g(p)$, *which contains* $A(p)$;

14. $r_g(p)$ - *the* **gravity radius** *of* p.

The geometric interrelations between the sets $A(p)$ and $A(p')$ of a polynomial p are invariant with respect to rotation, translation and scaling. Studying these interrelations, we may prescribe the circumself center and the circumself radius of the set $A(p)$, or the gravity center and the gravity radius of this set.

Definition 2.7 *Set the following notations:*

1. $\mathcal{P}$ *- the set of all polynomials* $p \in \mathcal{M}$ *with circumself disk* $D_c(p) = D(c_c(p), r_c(p))$ $= D(0,1) = D$;

2. $\mathcal{P}_n$ *- the set of all polynomials* $p \in \mathcal{P}$ *of degree* n;

3. $\mathcal{P}_{n,m}$; $m \leq n$ *- the set of all polynomials* $p \in \mathcal{P}_n$ *of the form* $p(z) = \prod_{k=1}^{m}(z - z_k)^{n_k}$, *where* $n = n_1 + n_2 + \cdots + n_m$ *and* $z_1, z_2, \ldots, z_m$ *are distinct.*

Lemma 2.6 *If* $p \in \mathcal{P}$, *then* $c_c(p') \in H(p) \subset D$.

Proof.On the contrary, suppose that $c_c(p') \notin H(p)$. Let l be a support line of $H(p)$, which separates $H(p)$ and $c_c(p')$. Let L be the half-plane defined by l, which contains $H(p)$. Then the subarc λ of the circle $C(c_c(p'), r_c(p')) = C_c(p')$ outside L has length larger than π. By Lemma 2.3, at least one zero of p' lies in λ, which is outside $H(p)$. This contradicts the Gauss–Lucas Theorem 2.1. $\square$

From Lemma 2.3 there follows:

Corollary 2.1 *If* $p \in \mathcal{P}_{n,m}$, *then:*

1. *At least two zeros of* p *lie in* C;

2. *At least one zero of* p *lies in each subarc of* C *with length* π, *therefore* $m \geq 2$.

Remark 2.1 *From* $h(A(p); A(q)) = 0$ *there follows only that* $A(p) = A(q)$. *If* $p \in \mathcal{M}_{m,m}$, $q \in \mathcal{M}_{n,n}$ *and* $h(A(p), A(q)) = 0$, *then* $m = n$ *and* $p = q$. *The set* $\mathcal{M}_n$ *is compact, but the set* $\mathcal{M}_{n,m}$ *is not compact.*

Definition 2.8 *Set the following notations*

1. $\mathcal{G}$ *- the set of all polynomials* $p \in \mathcal{M}$ *with gravity disk* $D_g(p) = D(c_g(p), r_g(p))$ $= D(0,1) = D$*;*

2. $\mathcal{G}_n$ *- the set of all polynomials* $p \in \mathcal{G}$ *of degree* n*;*

3. $\mathcal{G}_{n,m}$*;* $m \leq n$ *- the set of all polynomials* $p \in \mathcal{G}_n$ *of the form* $p(z) = \prod_{k=1}^{m}(z - z_k)^{n_k}$*, where* $n = n_1 + n_2 + \cdots + n_m$ *and* $z_1, z_2, \ldots, z_m$ *are distinct.*

From Lemma 2.5 it follows:

Corollary 2.2 *If* $p \in \mathcal{G}$*, then at least one zero of* p *lies in* C*.*

Lemma 2.7 *If* $p \in \mathcal{M}_{n,m}$*, then* $p' \in \mathcal{M}_{n-1,m'}$*, where* $m' = m - 1 + (n_1 - 2)_+ + (n_2 - 2)_+ \cdots + (n_m - 2)_+$ *and* $x_+ = 1$ *for* $x \geq 0$*,* $x_+ = 0$ *for* $x < 0$*.*

Proof. By Definition 2.7

$$p'(z) = n(z - \zeta_1)\cdots(z - \zeta_{n-1})\prod_{k=1}^{m}(z - z_k)^{n_k - 1} = np_1(z)q(z).$$

To prove the lemma, observe that $A(p_1) \bigcap A(q) = \emptyset$. Suppose that $\zeta_j = z_k$ for some $j \in \{1, 2, \ldots, m-1\}$ and $k \in \{1, 2, \ldots, m\}$. Then the multiplicity of z_k will be $> n_k$, which is a contradiction. $\square$

2.4 Basic statements

Theorem 2.1 (Gauss–Lucas Theorem). *For every algebraic polynomial* p*,* $A(p') \subset H(p)$*.*

Theorem 2.2 (Grace–Heawood, see M. Marden [53, p. 644]). *If* z_1 *and* z_2 *are any two distinct zeros of a polynomial* p *of degree* n*, then at least one critical point of* p *lies in the disk* $D((z_1 + z_2)/2, (|z_1 - z_2|/2)\cot(\pi/n))$*.*

From the process of proving the Grace–Heawood Theorem 2.2 there follows:

Lemma 2.8 *Let* z_1 *and* z_2 *be two different zeros of the polynomial* p *and* l *be the bisector of the segment from* z_1 *to* z_2*. Then in every two closed half-plane, defined by* l*, there is at least one zero of* p'*.*

The bisector property was noticed for the first time by G. Szegó [9].

Corollary 2.3 *If* $p(z)$ *is a polynomial and* $p(1) = p(-1)$*, then* $p'(z)$ *has at least one zero in each half-planes* $\{z : \operatorname{Re} z \geq 0\}$ *and* $\{z : \operatorname{Re} z \leq 0\}$*.*

Lemma 2.9 (Alexander, J. W. [1], Kakeya, S. [7]). *If $p(z)$ is a polynomial of degree n such that $p'(z) \neq 0$ in $D(0,R)$, then $p(z)$ is univalent in $D(0, R\sin(\pi/n))$.*

Definition 2.9 *The polynomials*

$$p(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are called **apolar** *if*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{2.5}$$

The equation (2.5) *is called the* **apolarity condition**.

Theorem 2.3 (Grace Apolarity Theorem, see M. Marden [55, p. 124]). *Let p and q be apolar, then any circular region containing all zeros of p or q contains at least a zero of the other.*

A circular region is a closed disk or a closed half-plain.

Theorem 2.4 (Walsh, see M. Marden [57, p. 644]). *If a polynomial p of degree n has m_1 zeros in a circle $C_1 = C(c_1, r_1)$ and the remaining $m_2 = n - m_1$ zeros in a circle $C_2 = C(c_2, r_2)$, then every critical point of p that does not lie in C_1 or in C_2 lies in a third circle $C_3 = C((m_2c_1 + m_1c_2)/n, (m_2r_1 + m_1r_2)/n)$.*

If in the Walsh Theorem c_2 is a single zero of p and $r_2 = 0$, we get

Corollary 2.4 *If a polynomial $p \in \mathcal{M}_n$ has $n-1$ zeros in a disk $D_1 = D(c,r)$ and a single zero a, then one critical point of p lies in the disk $D_3 = D((c+(n-1)a)/n, r/n)$.*

From a statement mentioned by M. Biernacki [3] and proved by W. Hayman [40, Theorem 6], there follows

Lemma 2.10 *Let $p \in \mathcal{P}_n$ and have $m+1$ nonzero terms and $\theta_2 - \theta_1 \geq 2\pi m/n$, then p has at least one zero $z_1 = re^{i\varphi}$ with $\varphi \in [\theta_1, \theta_2]$.*

3 Hausdorff deviations

For a polynomial p from $\mathcal{P}$ or $\mathcal{G}$, we shall consider the estimate from above of the deviation of one of the sets

$$A(p), H(p), D = D_c(p) = D_g(p) \tag{3.6}$$

from one of the sets

$$A(p'), H(p'), D_c(p'), D_g(p') \tag{3.7}$$

and vice versa. There are all together 24 such deviations. We shall consider also the nontrivial estimates from below, i. e., which are not equal to zero. Every deviation of a set (3.6) from a set (3.7) has a trivial estimate from below. Observe that if p has only multiple zeros, then $A(p) \subset A(p')$. Conversely, for a polynomial $p \in \mathcal{P}$, the set $A(p')$ is never a subset of $A(p)$.

From Definition 2.6, items 9 and 14, follows that

$$A(p') \subset H(p') \subset \begin{cases} D_c(p'), \\ D_g(p'). \end{cases} \tag{3.8}$$

3.1 Deviations of $A(p)$

3.1.1 Polynomials from $\mathcal{P}$

By the introduced notations, Conjecture 1 is formulated as follows.

Conjecture 1 *For every $p \in \mathcal{P}$, the following inequality holds:*

$$\rho(A(p); A(p')) \le 1$$

.

From (3.8) and Lemma 2.1 there follows that

$$\rho(A(p); A(p')) \ge \rho(A(p); H(p')) \ge \begin{cases} \rho(A(p); D_c(p')), \\ \rho(A(p); D_g(p')). \end{cases} \tag{3.9}$$

Then we formulate a weaker conjecture than Conjecture 1.

Conjecture 3 *For every polynomial $p \in \mathcal{P}$, the following inequalities hold:*

$$\left.\begin{array}{l} \rho(A(p); D_c(p')) \\ \rho(A(p); D_g(p')) \end{array}\right\} \le \rho(A(p); H(p')) \le 1. \tag{3.10}$$

The sharpness of the inequalities (3.10) is verified by the polynomial $p(z) = z^n - 1 \in \mathcal{P}$ for every $n \ge 2$. Moreover, in 1972 D. Phelps and R. S. Rodriguez formulated:

Conjecture 4 (Phelps–Rodriguez Conjecture [67]).
If $p \in \mathcal{P}_n(1)$ and $\rho(A(p); A(p')) = 1$, then $p(z) = z^n - 1$.

In other words, the polynomial $p(z) = z^n - 1$ is the only extremal polynomial in $\mathcal{P}_n(1)$ for the Conjecture 1. Conjecture 4 is proved for $n \le 8$ by J. E. Brown and G. Xiang [27].

Logically, the inequalities $\rho(A(p); D_c(p')) \le 1$ and $\rho(A(p); D_g(p')) \le 1$ are the weakest in (3.10) and may be the easiest to prove.

3.1.2 Polynomials from $\mathcal{G}$

For $p \in \mathcal{G}$, from (3.8) and Lemma 2.1 there follows that

$$\rho(A(p); A(p')) \geq \rho(A(p); H(p')) \geq \begin{cases} \rho(A(p); D_c(p')), \\ \rho(A(p); D_g(p')). \end{cases}$$

Theorem 3.1 *For every polynomial $p \in \mathcal{G}$ the following inequalities hold:*

$$\left. \begin{array}{l} \rho(A(p); D_c(p')) \\ \rho(A(p); D_g(p')) \end{array} \right\} \leq \rho(A(p); H(p')) \leq 1$$

Proof. As $p \in \mathcal{G}$, $c_g(p) = c_g(p') = 0$ and $0 \in H(p')$, therefore $\rho(A(p); H(p')) \leq 1$. □

We may formulate an analog of Conjecture 1:

Conjecture 5 *For every polynomial $p \in \mathcal{G}$, the inequality $\rho(A(p); A(p')) \leq 1$ holds.*

The sharpness of the Conjecture 5 is verified by the polynomial $p(z) = z^n - 1 \in \mathcal{G}$ for every $n \geq 2$. Logically, Conjecture 5 is weaker than Conjecture 1, as the gravity radius is larger or equal to the circumself radius of a polynomial. We may formulate Conjecture 5 as follows.

- Conjecture 1 is true for polynomials $p \in \mathcal{P}$ of the form $p(z) = z^n + a_2 z^{n-2} + \cdots + a_n$.

It is interesting to see if the following modification of Ratti–Schmeisser Conjecture 2 may be refuted by the method of M. J. Miller [60], see also S. Kumar and B. G. Shenoy [49].

Conjecture 6 *If $p \in \mathcal{G}$, then for every zero z_k of p, the disk $D(z_k/2, 1 - |z_k|/2)$ contains at least one zero of $p'(z)$.*

3.2 Deviations of $A(p')$

3.2.1 Polynomials from $\mathcal{P}$

By the introduced notations, the Aziz' Theorem 1.1 is formulated in the following form.

Theorem 1.1 (Aziz' Theorem). *For every $p \in \mathcal{P}$, the inequality $\rho(A(p'); A(p)) \leq 1$ holds.*

From (3.8) and Gauss–Lucas Theorem 2.1 there follows that for every $p \in \mathcal{P}$, the following equalities hold:

$$\rho(A(p'); D) = \rho(A(p'); H(p)) = 0.$$

The estimate of $\rho(A(p'); A(p))$ from below is not trivial, see **4.3.2**.

By Aziz' Theorem 1.1, the Conjecture 1 may be formulated in the following equivalent form.

Conjecture 1* *For every $p \in \mathcal{P}$, the inequality $h(A(p), A(p')) \leq 1$ holds.*

3.2.2 Polynomials from $\mathcal{G}$

From Aziz' Theorem 1.1 and Gauss–Lucas Theorem 2.1 there follows that for every $p \in \mathcal{G}$, the inequality $\rho(A(p'); A(p)) \leq 1$ and equalities $\rho(A(p'); D) = \rho(A(p'); H(p)) = 0$ hold.

3.3 Deviations of $H(p)$

3.3.1 Polynomials from $\mathcal{P}$

In 1977, G. Schmeisser formulated

Conjecture 7 (Schmeisser's Conjecture [79]). *For every $p \in \mathcal{P}$, the inequality $\rho(H(p); A(p')) \leq 1$ holds.*

Conjecture 7 is stronger than Conjecture 1, by Lemma 2.1, as $A(p) \subset H(p)$. From the proof of Lemma 5.1 it follows that Conjecture 7 is true for a polynomial p if $\hat{A}(p) = A(p)$, see Definition 2.3.

From (3.8) and Lemma 2.1

$$\rho(H(p); A(p')) \geq \rho(H(p); H(p')) \geq \begin{cases} \rho(H(p); D_c(p')), \\ \rho(H(p); D_g(p')) \end{cases}$$

holds. Then we formulate a weaker conjecture than Schmiesser's Conjecture 7.

Conjecture 8 *For every polynomial $p \in \mathcal{P}$, the inequalities*

$$\left.\begin{array}{l} \rho(H(p); D_c(p')) \\ \rho(H(p); D_g(p')) \end{array}\right\} \leq \rho(H(p); H(p')) \leq 1 \tag{3.11}$$

hold.

The sharpness of the inequalities (3.11) is verified by the polynomial $p(z) = z^n - 1 \in \mathcal{P}$ for every $n \geq 2$. Logically, the inequalities $\rho(H(p); D_c(p')) \leq 1$ and $\rho(H(p); D_g(p')) \leq 1$ are the weakest in (3.11) and may be the easiest to prove.

3.3.2 Polynomials from $\mathcal{G}$

For every $p \in \mathcal{G}$, from (3.8) and Lemma 2.1, the following inequalities hold:

$$\rho(H(p); A(p')) \geq \rho(H(p); H(p')) \geq \begin{cases} \rho(H(p); D_c(p')), \\ \rho(H(p); D_g(p')). \end{cases}$$

Theorem 3.2 *For every $p \in \mathcal{G}$, the following inequalities hold:*

$$\left.\begin{array}{l} \rho(H(p); D_c(p')) \\ \rho(H(p); D_g(p')) \end{array}\right\} \leq \rho(H(p); H(p')) \leq 1.$$

Proof. It is the same as the proof of the Theorem 3.1.
We may formulate an analog of Conjecture 7:

Conjecture 9 *If $p \in \mathcal{G}$, then the inequality $\rho(H(p); A(p')) \leq 1$ holds.*

The sharpness of Conjecture 9 is verified by the polynomial $p(z) = z^n - 1 \in \mathcal{G}$ for every $n \geq 2$. Logically, the Conjecture 9 is weaker than Conjecture 7, as the gravity radius is larger or equal to the circumself radius of a polynomial.

3.4 Deviations of $H(p')$

3.4.1 Polynomials from $\mathcal{P}$

The inequality $\rho(H(p'); A(p)) \leq 1$ does not follow directly from the Aziz' Theorem 1.1, as by Lemma 2.1, $\rho(H(p'); A(p)) \geq \rho(A(p'); A(p))$. We prove a little stronger theorem than the Aziz' Theorem 1.1.

Theorem 3.3 *If $p \in \mathcal{P}$, then the inequality $\rho(H(p'); A(p)) \leq 1$ holds.*

Proof. By the Gauss–Lucas Theorem 2.1 $A(p') \subset D$. Let $\zeta \in H(p')$. If $\zeta = 0$, then obviously $\rho(\zeta, A(p)) \leq 1$. Suppose that $\zeta \neq 0$ and consider the half-plane L_ζ, which is: 1) one of the two half-planes determined by the line l_ζ, which contains ζ and is orthogonal to the radius vector of ζ; 2) L_ζ does not contain the origin 0. From the Gauss–Lucas Theorem 2.1 there follows that at least one zero $z_k \in A(p)$ is in L_ζ, otherwise ζ will not be in $H(p')$. Then the dick $D(\zeta, 1)$ and the disk $D(2\zeta, 1)$ contain the set $L_\zeta \bigcap D$, in which there is at least one zero of p. $\square$

From Gauss–Lucas Theorem 2.1 it follows that $\rho(H(p'); D) = \rho(H(p'); H(p)) = 0$, as $H(p') \subset H(p) \subset D$.

3.4.2 Polynomials from $\mathcal{G}$

For every polynomial $p \in \mathcal{G}$, the inequality $\rho(H(p'); A(p)) \leq 1$ is proved as for $p \in \mathcal{P}$, see Theorem 3.3. The equalities $\rho(H(p'); D) = \rho(H(p'); H(p)) = 0$ follow from the Gauss–Lucas Theorem 2.1.

3.5 Deviations of D

3.5.1 Polynomials from $\mathcal{P}$

For every $p \in \mathcal{P}$, from (3.8) and Lemma 2.1, it follows that

$$\left.\begin{array}{l}\rho(D; D_c(p')) \\ \rho(D; D_g(p'))\end{array}\right\} \le \rho(D; H(p')) \le \rho(D; A(p')).$$

For the Hausdorff deviation $\rho(D; A(p'))$ we formulate

Conjecture 10 *For every polynomial $p \in \mathcal{P}$, the inequality*

$$\rho(D; A(p')) \le \sqrt{2} \tag{3.12}$$

holds.

The sharpness of the inequality (3.12) for polynomials of degree $n \ge 3$ is verified by the polynomial

$$p(z) = (z^2 + 1)(z - 1)^{n-2} \in \mathcal{P}. \tag{3.13}$$

The zeros of the derivative $p'(z) = (nz^2 - 2z + n - 2)(z + 1)^{n-3}$ are

$$\zeta_1 = (1 + i\sqrt{n^2 - 2n - 1})/n, \quad \zeta_2 = (1 - i\sqrt{n^2 - 2n - 1})/n \tag{3.14}$$

and $\zeta_j = 1$ for $j = 3, 4, \ldots, n-1$. A simple calculation gives

$$\rho(D; A(p')) \ge |1 + \zeta_1| = \sqrt{2}.$$

The Hausdorff deviation $\rho(D; H(p'))$ is larger than 1 for some polynomials $p \in \mathcal{P}$.

Conjecture 11 *For every polynomial $p \in \mathcal{P}$, the inequality*

$$\rho(D; H(p')) \le 1 + \frac{1}{n} \tag{3.15}$$

holds.

If true, Conjecture 11 is strict by the polynomial (3.13).

For the Hausdorff deviation $\rho(D; D_c(p'))$ we formulate an analog of Conjecture 1.

Conjecture 12 *For every polynomial $p \in \mathcal{P}$, the inequalities $\rho(D; D_c(p')) \le 1$ and $\rho(D; D_g(p')) \le 1$ hold.*

3.5.2 Polynomials from $\mathcal{G}$

Theorem 3.4 *For every $p \in \mathcal{G}$, the following inequalities hold:*

$$\left.\begin{array}{l}\rho(D; D_c(p')) \\ \rho(D; D_g(p'))\end{array}\right\} \leq \rho(D; H(p')) \leq 1.$$

Proof. As $p \in \mathcal{G}$, then $0 \in H(p')$ and

$$\rho(D; H(p')) \leq \rho(D; 0) = 1. \quad \square$$

Conjecture 13 *For every $p \in \mathcal{G}$, the inequality $\rho(D; A(p')) \leq 1$ holds.*

3.6 Deviations of $D_c(p')$

3.6.1 Polynomials from $\mathcal{P}$

Theorem 3.5 *If $p \in \mathcal{P}$, then the following inequalities hold:*

$$\rho(D_c(p'); D) \leq \rho(D_c(p'); H(p)) \leq 1 \quad \text{and} \quad \rho(D_c(p'); A(p)) \leq \sqrt{2}.$$

Proof. The sharpness of the first two inequalities follows by the polynomial $p(z) = z^n - 1$ and of the third, by the polynomial $p(z) = (z^2+1)^m$ for $m \geq 2$.

By definition $r_c(p') \leq 1$ and from Lemma 2.6 there follows that $c_c(p') \in H(p)$, since for every $\zeta \in D_c(p')$, we have $\rho(\zeta; H(p)) \leq 1$ and

$$\rho(D_c(p'); H(p)) = \sup\{\rho(\zeta; H(p)) : \zeta \in D_c(p')\} \leq 1.$$

The first two inequalities are proved. From the inequality $\rho(D_c(p'); H(p)) \leq 1$ and Lemma 2.4 $\rho(D_c(p'); A(p)) \leq \sqrt{2}$ follows. $\square$

3.6.2 Polynomials from $\mathcal{G}$

Theorem 3.6 *For every $p \in \mathcal{G}$, the following inequalities hold:*

$$\rho(D_c(p'); D) \leq \rho(D_c(p'); H(p)) \leq 1 \quad \text{and} \quad \rho(D_c(p'); A(p)) \leq \sqrt{2}$$

Proof. By Lemma 2.6 $c_c(p') \in H(p)$. By definition $r_c(p') \leq r_c(p) = 1$, hence $\rho(D_c(p'); D) \leq \rho(D_c(p'); H(p)) \leq 1$. The last equality follows from the previous one and Lemma 2.4. $\square$

3.7 Deviations of $D_g(p')$

3.7.1 Polynomials from $\mathcal{P}$

Theorem 3.7 *If $p \in \mathcal{P}$, then the following inequalities hold:*

$$\rho(D_g(p'); D_g(p)) \leq \rho(D_g(p'); H(p)) \leq 1 \quad \text{and} \quad \rho(D_g(p'); A(p)) \leq \sqrt{2}$$

Proof. The same as for Theorem 3.6. $\square$

3.7.2 Polynomials from $\mathcal{G}$

Theorem 3.8 *If $p \in \mathcal{G}$, then*

$$\rho(D_g(p'); D) = 0, \quad \rho(D_g(p'); H(p)) \le 1 \quad \text{and} \quad \rho(D_g(p'); A(p)) \le \sqrt{2}. \tag{3.16}$$

Proof. The equality in (3.16) follows from $D_g(p') \subset D$, as the two disks are concentric and $r_g(p') \le r_g(p) = 1$.

The inequality $\rho(D_g(p'); H(p)) \le 1$ follows from the fact that $c_g(p') \in H(p)$ and $r_g(p') \le 1$. The inequality $\rho(D_g(p'); A(p)) \le \sqrt{2}$ follows from Lemma 2.4 and the previous inequality. The sharpness of the last inequality follows by the polynomial $p(z) = (z^2 + 1)^m$ for $m \ge 2$. □

3.8 Summary of the estimations from above

To summarize the results, we present tables of the estimations from above of all Hausdorff deviations, considered in this section.

3.8.1 For polynomials from $\mathcal{P}$

	$A(p')$	$H(p')$	$D_c(p')$	$D_g(p')$
$A(p)$	1 Conj. 1	1 Conj. 3	1 Conj. 3	1 Conj. 3
$H(p)$	1 Conj. 7	1 Conj. 8	1 Conj. 8	1 Conj. 8
D	$\sqrt{2}$ Conj. 10	$1+1/n$ Conj. 11	1 Conj. 12	1 Conj. 12

	$A(p)$	$H(p)$	D
$A(p')$	1 Th. 1.2	0 Th. 2.1	0 Th. 2.1
$H(p')$	1 Th. 3.3	0 Th. 2.1	0 Th. 2.1
$D_c(p')$	$\sqrt{2}$ Th. 3.5	1 Th. 3.5	1 Th. 3.5
$D_g(p')$	$\sqrt{2}$ Th. 3.7	1 Th. 3.7	1 Th. 3.7

3.8.2 For polynomials from $\mathcal{G}$

	$A(p')$	$H(p')$	$D_c(p')$	$D_g(p')$
$A(p)$	1 Conj. 5	1 Th.; 3.1	1 Th.; 3.1	1 Th.; 3.1
$H(p)$	1 Conj. 9	1 Th. 3.2	1 Th. 3.2	1 Th. 3.2
D	$\sqrt{2}$ Conj. 13	1 Th. 3.4	1 Th. 3.4	1 Th. 3.4

	$A(p)$	$H(p)$	D
$A(p')$	1 Th. 1.2	0 Th. 2.1	0 Th. 2.1
$H(p')$	1 Th. 3.3	0 Th. 2.1	0 Th. 2.1
$D_c(p')$	$\sqrt{2}$ Th. 3.6	1 Th. 3.6	1 Th. 3.6
$D_g(p')$	$\sqrt{2}$ Th. 3.9	1 Th. 3.8	1 Th. 3.8

4 Extremal polynomials

Definition 4.1 *Let $F(p)$ be a continuous functional defined in a compact set $\mathcal{U}$ of polynomials p. Then there exists*

$$\overline{F}(\mathcal{U}) = \sup\{F(p):\ p \in \mathcal{U}\} \quad \text{and} \quad \underline{F}(\mathcal{U}) = \inf\{F(p):\ p \in \mathcal{U}\}.$$

The polynomial $p \in \mathcal{U}$ with $F(p) = \overline{F}(\mathcal{U})$ is called **supremal** *in $\mathcal{U}$ for F and the polynomial $p \in \mathcal{U}$ with $F(p) = \underline{F}(\mathcal{U})$ is called* **infimal** *in $\mathcal{U}$ for F.*

If we have to prove an estimate for a continuous functional F, defined in a compact set of polynomials $\mathcal{U}$ it is sufficient to consider only the respective supremal polynomials, if the estimate is from above, and only the respective infimal polynomials, if the estimate is from below.

In this section we consider extremal polynomials (supremal and infimal) in different compact sets of polynomials for different Hausrorff deviations, which are continuous functionals.

Definition 4.2 *For $S \in \{A, H, D\}$ and $T \in \{A, H, D_c, D_g\}$, define*
$\overline{\rho}(S(\mathcal{U}); T(\mathcal{U}')) = \sup\{\rho(S(p); T(p')) : p \in \mathcal{U}\}$,
$\underline{\rho}(S(\mathcal{U}); T(\mathcal{U}')) = \inf\{\rho(S(p); T(p')) : p \in \mathcal{U}\}$,
$\overline{\rho}(T(\mathcal{U}'); S(\mathcal{U})) = \sup\{\rho(T(p'); S(p)) : p \in \mathcal{U}\}$,
$\underline{\rho}(T(\mathcal{U}'); S(\mathcal{U})) = \inf\{\rho(T(p'); S(p)) : p \in \mathcal{U}\}$.

4.1 Supremal polynomials

In this section we consider some examples and properties of supremal polynomials.

4.1.1 $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}'_n))$

To prove the genegal form of Conjecture 1, it is sufficient to prove it for the supremal polynomials $p \in \mathcal{P}_n$ for $\rho(A(p); A(p'))$, or to find the value of $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}'_n))$. The superextremal polynomials in $\mathcal{P}_n$ for $\rho(A(p); A(p'))$ were defined by D. Phelps and R. S. Rodriguez (1972) [67] to formulate Conjecture 4.

The only polynomial $p \in \mathcal{P}_2(1)$ is $p(z) = z^2 - 1$, hence $\overline{\rho}(A(\mathcal{P}_2); A(\mathcal{P}'_2)) = 1$. The only subextremal polynomial $p \in \mathcal{P}_3(1)$ is $p(z) = z^3 - 1$, hence $\overline{\rho}(A(\mathcal{P}_3); A(\mathcal{P}'_3)) = 1$.

4.2 Maximal polynomials

M. J. Miller [60] studies special supremal polynomials. To formulate the main results of M. J. Miller, concerning the supremal polynomials for $\rho(A(p); A(p'))$, we need some notations.

Definition 4.3 (Miller, M. J. [60]). *A polynomial $p \in \mathcal{P}(b)$ is called* **maximal with respect to** *a, if for every $q \in \mathcal{P}(b)$, the inequality*

$$\rho(a; A(p')) \geq \rho(a; A(q'))$$

holds, and it is called **2-maximal in respect to** *a, if for every $q \in \mathcal{P}(b)$, the inequality*

$$\rho(a; A_p(p')) \geq \rho(a; A_q(q'))$$

holds. The circles $C(a, \rho(a; A(p')))$ and $C(a, \rho(a; A_p(p')))$ are called **critical circles**, *see Definition 2.6.* [4]

Theorem 4.1 (Miller, M. J. [60]). *Let $p(z) = (z-b)\prod_{k=1}^{m}(z-z_k)^{n_k}$ with distinct zeros $b, z_1, z_2, \ldots, z_m$, $|z_k| = 1$ for $k = 1, 2, \ldots, s$ and $|z_k| < 1$ for $k = s+1, s+2, \ldots, m$. Let $A_p(p') = \{\zeta_1, \zeta_2, \ldots, \zeta_m\}$, $|\zeta_j - a| = \rho(a; A_p(p'))$ for $j = 1, 2, \ldots, r$ and $|\zeta_j - a| > \rho(a; A_p(p'))$ for $j = r+1, r+2, \ldots, m$. If $|b| < 1$, p is 2-maximal with respect to a and $p'(z)/\prod_{k=1}^{m}(z-z_k)^{n_k-1}$ has no multiple zeros on the critical circle $C(a, \rho(a; A_p(p')))$, then*

$$2r + s \geq m + 2 \quad \text{and} \quad r \geq \frac{1}{6}(m+2).$$

Theorem 4.2 (Miller, M. J. [61]). *Suppose that the polynomial $p \in \mathcal{P}_n(b)$ is maximal in respect to a. Let s be the number of the zeros of p (counting multiplicities) on C and r be the number of the zeros of p (counting multiplicities) on $C(a, \rho(a; A(p))$. Then $2r + s \geq n + 1$.*

[4] The example in the last line of page 288 in [60] is not correct.

.

Conjecture 14 (Miller, M. J. [60]). *If $p \in \mathcal{P}_n(b)$ is maximal with respect to a, then*

(1) all zeros of p' lie on the circle $C(a, \rho(a; A(p')))$ and, given this, as many zeros of p as possible, lie on the unit circle C;

(2) if a and b are real and p is monic, then p is a real polynomial.

By "as many zeros of p as possible", it is meant that p has l zeros on the unit circle, where l is the largest integer such that there exists a polynomial $q \in \mathcal{P}_n(b)$ with all zeros of q' on a circle centered at a and l zeros of q on the unit circle.

In Miller, M. J. (1997) [63] examples contradicting the second part of Conjecture 14 for $p \in \mathcal{P}_6(b)$ are given.

4.2.1 $\overline{\rho}(A(\mathcal{P}'_n); A(\mathcal{P}_n))$

We know that $\overline{\rho}(A(\mathcal{P}'_n); A(\mathcal{P}_n)) = 1$, by the Aziz' Theorem 1.1 and the supremal polynomial $p(z) = z^n - 1$ in this case. It is interesting to mention that if $n = 2m$ is even, then $p(z) = z^n - 1$ is not the unique supremal polynomial in $\mathcal{P}_n(1)$ for $\rho(A(p'); A(p))$, as one may expect as in analogy with Conjecture 4. Another supremal polynomial in $\mathcal{P}_{2m}(1)$ for $\rho(A(p'); A(p))$ is $q(z) = (z^2 - 1)^m$.

4.2.2 $\overline{\rho}(D; A(\mathcal{P}'_n))$

It is obvious that $\overline{\rho}(D; A(\mathcal{P}'_2)) = 1$. For $n \geq 3$, by polynomial (3.13) we have

$$\overline{\rho}(D; A(\mathcal{P}'_n)) \geq \sqrt{2},$$

see Conjecture 10. We may expect that the polynomial (3.13) is supremal in $\mathcal{P}_n$ for $\rho(D; A(p'))$, as it fulfills the hypothesis of Conjecture 14 for a supremal polynomial. All the zeros of this polynomial lie on the unit circle C and all its critical points lie on the critical circle $C(-1, \sqrt{2})$.

4.3 Infimal polynomials

4.3.1 $\underline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}'_n))$

It is easy to see that

$$\underline{\rho}(A(\mathcal{P}_2); A(\mathcal{P}'_2)) = 1, \quad \underline{\rho}(A(\mathcal{P}_3); A(\mathcal{P}'_3)) = \frac{2}{3} \quad \text{and} \quad \underline{\rho}(\mathcal{P}_n; \mathcal{P}'_n) = 0$$

for $n \geq 4$.

The set $\mathcal{P}_2(1)$ has only one element, the polynomial $p(z) = z^2 - 1$. There are two infimal polynomials in $\mathcal{P}_3(1)$ for $\rho(A(\mathcal{P}_3); A(\mathcal{P}'_3))$, the polynomials $p(z) = (z^2 + 1)(z - 1)$ and $-p(-z)$. For $n \geq 4$ every polynomial $p \in \mathcal{P}_n$ with only multiple zeros is infimal polynomial in $\mathcal{P}_n$ for $\rho(A(p); A(p'))$.

4.3.2 $\underline{\rho}(A(\mathcal{P}_n'); A(\mathcal{P}_n))$

It is trivial that $\underline{\rho}(A(\mathcal{P}_2'); A(\mathcal{P}_2)) = 1$. Simple calculation gives $\underline{\rho}(A(\mathcal{P}_3'); A(\mathcal{P}_3)) = 1/\sqrt{3}$. By the polynomial

$$p(z) = \prod_{k=1}^{n} \left(z + 1 - \frac{2(k-1)}{n-1} \right) \tag{4.17}$$

and the Rolle's Theorem, we get

$$\underline{\rho}(A(\mathcal{P}_n'); A(\mathcal{P}_n)) \le \frac{2}{n-1}.$$

The polynomial (4.17) is far from infimal. By the Chebyshev polynomial

$$p(x) = \cos\left(\arccos\left(x \cos\frac{\pi}{2n}\right)\right)$$

we have

$$\underline{\rho}(A(\mathcal{P}_n'); A(\mathcal{P}_n)) \le \tan\frac{\pi}{2n} \approx \frac{\pi}{2n} < \frac{2}{n-1}.$$

Problem 1 *Find* $\underline{\rho}(A(\mathcal{P}_n'); A(\mathcal{P}_n))$.

The examples considered above give motivation for

Conjecture 15 *If* $\underline{\rho}(T(\mathcal{P}_n'(1)); S(\mathcal{P}_n(1))) > 0$ (see Definition 4.1), *then every infimal polynomial for it has real coefficients and real zeros.*

Remark 4.1 *The polynomial* $p(z) = z^n - 1$ *is supremal for many functionals considered in Section 3. This polynomial has the following properties:*

1. $p \in \mathcal{P}_n \bigcap \mathcal{G}_n \bigcap \mathcal{P}_{n,n} \bigcap \mathcal{G}_{n,n}$;
2. $D_c(p) = D_g(p)$;
3. *The measure* $\mu(H(p))$ *is as large as possible.*
4. *The measure* $\mu(H(p'))$ *is as small as possible.*

It is natural to expect that every other extremal pair will have similar properties.

5 Statements for a particular zero

Conjecture 1 and other conjectures of Hausdorff-type involve all the zeros of a polynomial. There are many statements, concerning Hausdorff-type conjectures, which verify these conjectures for a particular zero. In this section we consider such statements.

Definition 5.1 *A zero z_k of $p \in \mathcal{P} \bigcap \mathcal{G}$ is called* **normal**, *if there exists a zero ζ_j of p', such that $|z_k - \zeta_j| \leq 1$.*

In other words, a zero is normal if Conjecture 1 is true for it. It is obvious that if $p \in \mathcal{P} \bigcap \mathcal{G}$ and $p'(0) = 0$, then 0 is a normal zero of p.

Theorem 5.1 *If $p \in \mathcal{P}_n(a) \bigcap \mathcal{G}_n(a)$ and $|p'(a)| \leq n$, then a is a normal zero of p.*

Proof. The polynomial $q(z) = nz^{n-1} + (-1)^n p'(a)$ is apolar (see Definition 2.9) to $p'(a - z) = (-)^n nz^{n-1} + \cdots - p'(a)$. If $|p'(a)| \leq n$, then all zeros of $q(z)$ lie in D. By Theorem 2.3 at least one zero of $p'(z + a)$ lies in D, or at least one critical point of $p(z)$ lies in $D(a, 1)$. $\square$

Remark 5.1 *If in Theorem 5.1 $|p'(a)| < n$, then there is a zero ζ_1 of p', such that $|\zeta_1 - a| < 1$.* See Z. Rubinstein [75, Theorem 3].

Using more complicated apolar polynomials to the polynomial $p'(z+a)$, P. Pflug and G. Schmieder (1994) [66] proved the following.

Theorem 5.2 (P. Pflug and G. Schmieder [66]). *Let $p \in \mathcal{P}_n(a)$ and*

$$\sigma_k = \sum_{2 \leq j_1 < \cdots < j_k \leq n} (a - z_{j_1}) \cdots (z - z_{j_k}); \quad k = 1, 2, \ldots, n-1;$$

$$\sigma_0 = 1; \quad \beta_k = \frac{k+1}{\binom{n-1}{k}} \sigma_{n-1-k}; \quad k = 0, 1, \ldots, n-1.$$

Then if there exists some $\zeta \in D$, such that for some $\nu \in \{3, 4, \ldots, n\}$

$$|\zeta \beta_1 + \beta_0| \leq |\zeta \beta_{\nu-1} + \beta_{\nu-2}|$$

holds, then a is a normal zero of p.

5.1 A zero in the unit disk

Lemma 5.1 (A. G. Goodman, Q. I. Rahman and J. Ratti[39], G. Schmeisser [77]). *If $p \in \mathcal{P}(1)$, then there exists $\zeta_1 \in A'(p)$ such that $\zeta_1 \in D(1/2, 1/2)$.*

Proof. If $p'(1) = 0$, then set $\zeta_1 = 1$. Let $p'(1) \neq 0$ and $p(z) = (z-1)q(z)$, then $p''(1)/p'(1) = 2q'(1)/q(1)$ and

$$\sum_{j=1}^{n-1} \frac{1}{1-\zeta_j} = 2\sum_{k=2}^{n} \frac{1}{1-z_k}. \tag{5.18}$$

Supposing $\Re(1/(1-\zeta_1)) = \max\{\Re(1/(1-\zeta_j)) : \ j = 1, 2, \ldots, n-1\}$, then from (5.18) we get

$$(n-1)\Re\left(\frac{1}{1-\zeta_1}\right) \geq 2\sum_{k=2}^{n} \Re\left(\frac{1}{1-z_k}\right) \geq n-1, \tag{5.19}$$

as $|z_k| \leq 1;\ k = 2, 3, \ldots, n$.

Hence $\Re(1/(1-\zeta_1)) \geq 1$, or $|\zeta_1 - 1/2| \leq 1/2$. $\square$

Corollary 5.1 *If $p \in \mathcal{P}(a) \bigcap \mathcal{G}(a)$ and $|a| = 1$, then a is a normal zero of p.*

In 1968, Z. Rubinstein proved a very important theorem for Conjecture 1.

Theorem 5.3 (Z. Rubinstein [75]). *If $p \in \mathcal{P}_n(1) \bigcap \mathcal{G}(1)$ and $p(z) \neq z^n - 1$, then there is a zero ζ_1 of p', such that $|\zeta_1 - 1| < 1$.*

Proof. Suppose the contrary, that $p'(z+1) \neq 0$ in the interior of D. Then, according to the well known Theorem of J. Dieudonné (1934) [4]

$$p'(z+1) = p'(1)(1 - zf(z))^{n-1} \quad \text{for} \quad z \in D, \tag{5.20}$$

where $f(z)$ is analytic and $|f(z)| \leq 1$ in D. From $p(z) \neq z^n - 1$ it follows that $f(z)$ is not the constant 1 and since $|f(0)| < 1$.

From (5.20) we get

$$p''(1) = (1-n)p'(1)f(0). \tag{5.21}$$

If $p(z) = (z-1)q(z)$, then $p'(1) = q(1)$ and $p''(1) = 2q'(1)$. Then from (5.21) we have

$$\frac{p''(1)}{p'(1)} = 2\frac{q'(1)}{q(1)} = 2\sum_{k=2}^{n} \frac{1}{1-z_k} = (1-n)f(0)$$

and

$$\left|\sum_{k=2}^{n} \frac{1}{1-z_k}\right| < \frac{n-1}{2}. \tag{5.22}$$

On the other hand, for $|z_k| \leq 1$ we have $\Re(1/(1-z_k)) \geq 1/2$ and

$$\left|\sum_{k=2}^{n} \frac{1}{1-z_k}\right| \geq \Re\left(\sum_{k=2}^{n} \frac{1}{1-z_k}\right) = \sum_{k=2}^{n} \Re\left(\frac{1}{1-z_k}\right) \geq \frac{n-1}{2},$$

which contradicts (5.22). □

In 1986 Q. M. Tariq generalized Lemma 5.1 for a multiple zero.

Theorem 5.4 (Q. M. Tariq [87]). *If $p(z) = (z-1)^k q(z)$, $q \in \mathcal{P}$ and $q(1) \neq 0$, then p' has at least k zeros in the disk $D(1/(k+1), k/(k+1))$.*

Theorem 5.5 (Q. M. Tariq [88]). *Let $p(z) = (z-a)^k q(z)$, $0 \leq a \leq 1$, $q \in \mathcal{P}_{n-k}$ and $k < n \leq (k+1)^2$, then p' has at least k zeros in the disk $D(a/(k+1), k/(k+1)) \subset D(a, 2k/(k+1))$.*

In 1987 Q. I. Rahman and Q. M. Tariq [70] formulated

Problem 2 (Q. I. Rahman and Q. M. Tariq [70]). *Find the smallest $\rho_{n,k}(a)$, such that for every polynomial $p \in \mathcal{P}_n$ of the form $p(z) = (z-a)^k q(z)$, at least k zeros of p' lie in the disk $D(a, \rho_{n,k}(a))$.*

From Theorem 5.5 follows that $\rho_{n,k}(a) \leq 2k/(k+1)$ for $k < n \leq (k+1)^2$. In Q. I. Rahman and Q. M. Tariq [70] some estimates for $\rho_{n,k}(a)$ are proved.

5.2 A zero close to the unit disk

In V. Vâjâtu and A. Zaharescu (1993) [93], M. J. Miller (1993) [62] and T. J. McCoy (1998) [58], zeros sufficiently close to the unit disk are considered.

Let $\tilde{\mathcal{P}}_n(a)$ be the set of all polynomials $p \in \mathcal{P}_n(a)$, $0 \leq a < 1$, such that $\rho(a; A(p')) \geq a$. M. J. Miller [62] observe that if $p \in \tilde{\mathcal{P}}_n(a)$ and a is close to 1, then all zeros of p' are close to 0. This follows from the fact that any sequence of polynomials $p_k \in \tilde{\mathcal{P}}_n(a_k)$, $a_k < 1$ and $\lim_{k\to\infty} a_k = 1$ has by compactness a subsequence, which converges to a polynomial $p^* \in \mathcal{P}_n(1)$ with $\rho(1; A(p')) = 1$. By Theorem 5.3 this polynomial must be $p^*(z) = z^n - 1$, so the zeros of the p_k' tend to the zeros of the derivative of p^*, which are all 0.

Theorem 5.6 (M. J. Miller [62]). *For every natural number n, there is a constant $k_n > 0$ such that for every $p \in \mathcal{P}_n(a)$, $0 < a < 1$, if a is sufficiently close to 1, then $\rho(a; A(p')) \leq 1 - k_n(1-a)$. Furthermore one can choose the k_n so that $\lim_{n\to\infty} k_n = 1/3$.*

From Theorem 5.6 it follows that if a zero is sufficiently close to the unit disk, then it is a normal zero.

Q. I. Rahman (1972) [69] studied the polynomials of the form $q(z) = (z-a)p(z)$, where $p \in \mathcal{P}_n$ and $|a| > 1$.

Theorem 5.7 (Q. I. Rahman [69]). *If $q(z) = (z-a)p(z)$, $p \in \mathcal{P}_n$ and $|a| > 1 + 2/n$, then q' has one and only one zero in the disk $D(a, (|a|+1)/(n+1))$. The remaining $n-1$ zeros of q' lie in D.*

The proof of Theorem 5.7 follows from Corollary 2.4.

5.3 A zero close to the origin

The fact that $z_1 = 0$ is a normal zero of every polynomial $p \in \mathcal{P}(0)$, motivates the following.

Problem 3 (S. Vernon [92]). *Find the biggest possible δ_n such that if $p \in \mathcal{P}_n(a)$ and $|a| \le \delta_n$, then a is a normal zero of p.*

If $\delta_n = 1$ for every natural $n \ge 2$, then Conjecture 1 is true.

Theorem 5.8 (S. Vernon [92]).

$$\delta_n \ge \frac{1}{2}\sqrt{\frac{5}{4} - n^{1/(n+1)}}.$$

The estimate for δ_n in Theorem 5.8 tends to 0 for $n \to \infty$. It is essential to find a $\delta > 0$, such that $\delta_n \ge \delta$ for every natural $n \ge 2$.

6 Polynomial sets for which Conjecture 1 is true

In this section we review some results for proofs of partial cases of Conjecture 1. We start with polynomials of restricted degree.

6.1 Polynomials of degree ≤ 8

All found results are listed by dates in **10.1**. Here we present some results and comments.

6.1.1 Polynomials of degree 3 and 4

There are a number of proofs of Conjecture 1 for polynomials of degree 3, see **10.1**, item 1, and of degree 4, see **10.1**, item 2. Here we present simple proofs of these cases. For polynomials of degree 2, Conjecture 1 is trivial.

Theorem 6.1 *Conjecture 1 is true for every polynomial $p \in \mathcal{P}_3(1)$.*

Proof. Let $p(z) = (z-1)(z-z_2)(z-z_3)$. If $|z_2| = |z_3| = 1$, then the proof follows from Corollary 5.1. If $|z_3| < 1$, then $z_2 = -1$ and

$$|p'(z_3)| = |(z_3 - 1)(z_3 + 1)| = |z_3^2 - 1| \leq 2.$$

Hence, the proof follows from Theorem 5.1 and Corollary 5.1. $\square$

Theorem 6.2 *Conjecture 1 is true for every polynomial $p \in \mathcal{P}_4(1)$.*

Proof. Let $p(z) = (z-1)(z-z_2)(z-z_3)(z-z_4)$.

1) Let $|z_2| = |z_3| = |z_4| = 1$, then the proof follows from Corollary 5.1.

2) Let $|z_2| = |z_3| = 1$, $z_2 = e^{i\varphi_2}$, $z_3 = e^{i\varphi_3}$, $0 \leq \varphi_2 \leq \pi$, $\pi \leq \varphi_3 \leq 2\pi$ and $z_4 = r_4 e^{i\varphi_4}$. If $0 \leq \varphi_4 \leq \varphi_2$, then $|(z_4 - 1)(z_4 - z_2)| \leq 2$, if $\varphi_2 \leq \varphi_4 \leq \varphi_3$, then $|(z_4 - z_2)(z_4 - z_3)| \leq 2$, if $\varphi_3 \leq \varphi_4 < 2\pi$, then $|(z_4 - z_3)(z_4 - 1)| \leq 2$, hence

$$|p'(z_4)| = |(z_4 - 1)(z_4 - z_2)||z_4 - z_3| \leq 2 \cdot 2 = 4$$

and the proof follows from Theorem 5.1 and Corollary 5.1.

3) Let $|z_2| = 1$, $|z_3| < 1$ and $|z_4| < 1$, then $z_2 = -1$ and

$$|p'(z_3)| = |z_3^2 - 1||z_3 - z_4| \leq 2 \cdot 2 = 4,$$

$$|p'(z_4)| = |z_4^2 - 1||z_4 - z_3| \leq 2 \cdot 2 = 4.$$

Hence, the proof follows from Theorem 5.1 and Corollary 5.1. $\square$

In a recent paper J. E. Brown and G. Xiang (1999) [27] proved

Theorem 6.3 (J. E. Brown and G. Xiang [27]) *Conjecture 1 is true for every polynomial $p \in \mathcal{P}_{n,m}$, if $m \leq 8$.*

The proof is very elaborate and is based on obtaining good upper and lower estimates on the product of the moduli of the critical points of p. The next step to $n = 9$ is possible by this method through very scrupulous calculations.

Conjecture 16 *Conjecture 1 is true for every polynomial of the form*
$q(z) = (z - z_1)^{m_1}(z - z_2)^{m_n} \cdots (z - z_n)^{m_n}$,
if it is true for $p(x) = (z - z_1)(z - z_2) \cdots (z - z_n)$.

6.2 The polynomial set $\mathcal{P}(0)$

In 1969 G. Schmeisser [77] proved the following statement.

Theorem 6.4 *Conjecture 1 is true for every polynomial $p \in \mathcal{P}(0)$.*

Proof. An elegant geometric proof is given in G. Gacs (1971) [36], based on Lemma 2.8. Let $z_1 \neq 0$ be a zero of $p \in \mathcal{P}(0)$ and l be the bisector of the segment from z_1 to 0. Let L be the half-plane defined by l and $0 \notin L$. Then by Lemma 2.8 and Gauss–Lucas Theorem 2.1 it follows that $(L \bigcap D) \subset D(z_1, 1)$ contains at least one zero of p'. $\square$

Remark 6.1 *Conjecture 1 is trivially true for every $p \in \mathcal{P}$, if $p'(0) = 0$.*

6.3 Polynomials invariant under rotation

In J. Dronka (1989) [33] it is proved that Conjecture 1 is true for a polynomial p, if it is invariant under a nontrivial transformation $w(z) = az + b$, i. e., $p(w(z)) = p(z)$. A polynomial $p \in \mathcal{P}$ may be invariant under the transformation $w(z)$ if $|a| = 1$ and $b = 0$. Let $p \in \mathcal{P}$ and for $|a| = 1, a \neq 1$ we have $p(az) = p(z)$. Then $ap'(az) = p'(z)$ and $ap'(0) = p'(0)$, therefore $p'(0) = 0$. Then Conjecture 1 is trivially true for this polynomial. Note that $p(z) = z^n - 1$ is invariant under the rotation $w(z) = az$ for $a = e^{2k\pi i/n}$; $k = 1, 2, \dots, n-1$.

6.4 Lacunary polynomials

Let $p \in \mathcal{P}_n$ and have the form

$$p(z) = \sum_{k=0}^{m} a_k z^{n_k}; \quad n = n_m > n_{m-1} > \cdots > n_0. \tag{6.23}$$

If $m < n$, the polynomial (6.23) is lacunary.

Theorem 6.5 (G. Schmeisser [79, Theorem 2]). *Conjecture 1 is true for the polynomial* (6.23)*, if* $n \geq 3m - 2$.

Proof. By Theorem 6.4 and Remark 6.1, we may assume that $n_0 = 0$ and $n_1 = 1$. Then Lemma 2.10 in conjunction with Gauss–Lucas Theorem 2.1 yields that for every $\alpha \in [0, 2\pi)$ the derivative $p'(z)$ has at least one zero in the sector $s(\varphi) = \{z : \varphi - \pi/3 \leq \arg z \leq \varphi + \pi/3\} \bigcap D$. Observe that for every $z_1 = re^{i\varphi} \in D$, we have $s(\varphi) \subset D(z_1, 1)$. □

Theorem 6.6 (G. Schmeisser [79, Theorem 3]). *Conjecture 1 is true for* $p \in \mathcal{P}_n$*, if*

$$p(z) = z^n + a_2 z^{n_2} + a_1 z^{n_1} + a_0 z^{n_0}, \quad \text{where} \quad n > n_2 > n_1 > n_0.$$

Proof. If $n \geq 7$, Theorem 6.5 applies. If $n \leq 8$, Theorem 6.3 applies.

6.5 Polynomials with $A(p) = \hat{A}(p)$

Theorem 6.7 *Conjecture 1 is true for all polynomials* $p \in \mathcal{P}$*, if* $A(p) = \hat{A}(p)$.

Proof. If all the corners of $A(p)$ for a polynomial p lie in the unit circle, then by Lemma 5.1 for every zero $|z_k|$ of p with $|z_k| = 1$, the disk $D_k = D(z_k/2, 1/2)$ contains at least one zero of p'. If $\hat{A}(p) = \{z_1, z_2, \dots, z_m\}$, $|z_k| = 1$; $k = 1, 2, \dots, m$, are all the corners of $A(p)$, then $A(p) \subset \bigcup_{k=1}^{m} D_k$. □

6.6 Polynomials p with given shape of $H(p)$

6.6.1 $H(p)$ is a segment

D. Phelps and R. S. Rodriguez (1972) [67] proved Conjecture 1 for all polynomials p with $H(p)$ a segment. If for a polynomial $p \in \mathcal{P}(1)$, the convex hull $H(p)$ is a segment, then p has only real zeros and real coefficients. This case is considered in **8.2**.

6.6.2 $H(p)$ is a triangle

In 1977 G. Schmeisser [79] proved Conjecture 1 for polynomials p with a convex hull $H(p)$, which is a triangle. The proof is purely geometric.

Conjecture 17 (Schmeisser, G [79]). *Conjecture 1 is true for all $p \in \mathcal{P}$ for which $H(p)$ is a quadrangular.*

7 Weaker estimates for $\rho(A(p); A(p'))$

In 1972, G. Schmeisser [78, p. 406, Problem 2] formulated

Problem 4 (G. Schmeisser [78]). *Find a constant ρ (as small as possible), such that for every $p \in \mathcal{P}_n$, the inequality*

$$\rho(A(p); A(p')) \leq \rho \tag{7.24}$$

holds.

From Gauss–Lucas Theorem 2.1 there follows that $\rho \leq 2$. In G. Schmeisser (1971) [78, p. 413, Satz 2] it is proved that $\rho \leq 1.568$. D. G. Dimitrov (1983) [32] proved that for $p \in \mathcal{P}_n(a)$, the inequality

$$\rho(A(p); A(p')) \leq 1 + \frac{n-1}{n}|a|$$

holds, from which Teorem 6.4 follows.

In 1982 M. Marden [55, p. 124, Theorem 1] proved

Theorem 7.1 (M. Marden [55]). *If $p \in \mathcal{P}_n$ and $M = \sup\{|p(z)| : |z| = 1\}$, then the inequality*

$$\rho(A(p); A(p')) \leq M^{1/(n-1)} \tag{7.25}$$

holds.

M. MARDEN [55, p. 126] claims that Theorem 7.1 validates Conjecture 1 "*in the general case when* $n \to \infty$".

The best estimate for ρ till now is given in BOJANOV, B. B., Q. I. RAHMAN AND J. SZYNAL (1985) [15], which is based on the following

Theorem 7.2 (B. D. Bojanov, Q. I. Rahman and J. Szynal [15]). *For every* $p \in \mathcal{P}_n$, *the inequality*

$$\rho(A(p); A(p')) \leq \rho_n = (1 + |z_1 z_2 \cdots z_n|)^{1/n}$$

holds.

After J. E. BROWN AND G. XIANG (1999) [27], see Theorem 6.3, proved Conjecture 1 for $n \leq 8$, from Theorem 7.2 there follows

Corollary 7.1 (B. D. Bojanov, Q. I. Rahman and J. Szynal). *For every* $p \in \mathcal{P}$, *the inequality*

$$\rho(A(p); A(p')) \leq 1.08006\ldots$$

holds.

We have also

Corollary 7.2 (Bojanov, B. D., Q. I. Rahman and J. Szynal). *For every* $p \in \mathcal{P}$, *the equality*

$$\lim_{n\to\infty} \rho(A(p); A(p')) = 1$$

holds.

In other words, Conjecture 1 is asymptotically true. In N. KJURKCHIEV (1992) [47], a variant of Theorem 7.2 is proved.

7.1 Proof of Theorem 7.2

We start with

Lemma 7.1 (B. D. Bojanov, Q. I. Rahman and J. Szynal [15]). *If* $q(z)$ *is a monic polynomial of degree* n, $q(0) = 0$ *and* $q'(0) \neq 0$ *in* $D(0, R)$, *then* $|q(z)| \geq R^n - (R - \lambda)^n$ *for* $|z| = \lambda \leq R \sin(\pi/n)$.

Proof. As q is monic

$$|q'(z)| \geq n(R - |z|)^{n-1} \quad \text{for} \quad |z| \leq R. \tag{7.26}$$

By Lemma 2.9 q is univalent in $D(0, R\sin(\pi/n))$. Let $\min\{|q(z)| : |z| = \lambda\} = |q(z_0)|$ and let $\varphi : z = z(t);\ 0 \le t \le 1$ be the pre-image in $D(0, R\sin(\pi/n))$ of the straight-line $[0, q(z_0)]$ under the mapping $q(z)$. Since $\arg(q(z(t)))$ is constant for $0 \le t \le 1$, we have

$$|q(z_0)| = \int_0^1 |q'(z(t))||z'(t)|\, dt.$$

Using (7.26) we get

$$|q(z_0)| > \int_0^1 (R - |z(t)|)^{n-1}|z'(t)|dt \ge n\int_0^1 (R - |z(t)|)^{n-1}\, d|z'(t)|$$
$$= R^n - (R-\lambda)^n. \qquad \square$$

Suppose that $\rho(A(p); A(p')) > (1 + |z_1 z_2 \cdots z_n|)^{1/n} = \rho_n$. Then there exists a zero z_ν such that $p'(z) \ne 0$ in $D(z_\nu, \rho_n)$. It is obvious that $|z_\nu| > \rho_n - 1$, otherwise $D \subset D(z_\nu, \rho_n)$, which contradicts $p'(z_\nu) \ne 0$ in $D(z_\nu, \rho_n)$. By Lemma 7.1 for the polynomial $q(z) = p(z - z_\nu)$, with $R = \rho_n$ and $\lambda = R - 1$, we obtain $|p(z - z_\nu)| > |z_1 z_2 \cdots z_n|$ for $|z - z_\nu| = \rho_n - 1$. Therefore $p(z)$ must assume the value $(-1)^n z_1 z_2 \cdots z_n = p(0)$ in some point z^* in $D(z_\nu, \rho_n - 1)$. Now, using Corollary 2.3, we conclude that $p'(z)$ has at least one zero ζ_j in the half-plane $L = \{z :\ |z - z^*/2| \le |z|\}$.[5] It is geometrically evident that $|\zeta_j - z^*| \le 1$ and $|\zeta_j - z_\nu| \le \rho_n$. That is a contradiction. $\square$

7.2 Proof of Corollary 7.1

The following statement is due to G. Schmeisser (1969) [77].

Lemma 7.2 (G. Schmeisser [77]) *If $p \in \mathcal{P}_n(a)$, $0 \le a \le 1$ and $p'(a) \ne 0$, then*

$$\Re\left(\frac{p''(1)}{p'(1)}\right) \ge \frac{(n-1)a_2}{a_1},$$

where

$$a_1 = n + 1 - a(n-1), \quad a_2 = n + 2 - a(n-2). \tag{7.27}$$

Proof. If

$$r_m e^{i\theta_m} = -\frac{1}{1 - z_m} + \sum_{k=1}^{n} \frac{1}{1 - z_k}; \quad m = 1, 2, \ldots, n. \tag{7.28}$$

then

$$\frac{p''(1)}{p'(1)} = \sum_{m=1}^{n} \frac{1}{1 - z_m + r_m^{-1} e^{-i\theta_m}}$$

and

$$\Re\left(\frac{p''(1)}{p'(1)}\right) = \Re(1 - a + r_1^{-1}e^{-i\theta_1})^{-1} + \sum_{m=2}^{n} \Re(1 - z_m + r_m^{-1}e^{-i\theta_m})^{-1}. \tag{7.29}$$

[5] In [15, p. 283] there are typographical errors. On p. 283 l. 4+, instead of $p(z-a)$ is $p(z+a)$ and on the same page l. 8+, instead of $|z - z^*/2|$ is printed $|z - z^*|$.

From (7.28) we have

$$r_1 \cos\theta_1 = \sum_{k=2}^{n} \Re\left(\frac{1}{1-z_k}\right) \geq \frac{n-1}{2} \tag{7.30}$$

and

$$r_m \cos\theta_m \geq \frac{1}{1-a} + \frac{n-2}{2}; \quad m = 2, 3, \ldots, n. \tag{7.31}$$

From (7.30) and (7.31) it follows that
$\mathrm{Re}(1 - a + r_1^{-1} e^{-i\theta_1})^{-1} \geq (n-1)/(n+1-a(n-1))$,
$\mathrm{Re}(1 - z_m + r_m^{-1} e^{-i\theta_m})^{-1} \geq (n - a(n-2))/2(n+1-a(n-1))$,
which with (7.29) completes the proof. □

If $p'(z) \neq 0$ in the disk $D((a+1)/a_2, a_1/a_2)$, where $a_1 = n+1-a(n-2)$, $a_2 = n+2-a(n-2)$, having the interval $[1-2a_1/a_2, 1]$ as a diameter, then by (5.19), we have

$$\Re\left(\frac{p''(1)}{p'(1)}\right) = \sum_{j=1}^{n-1} \Re\left(\frac{1}{1-\zeta_j}\right) < \sum_{j=1}^{n-1} \frac{a_2}{2a_1} = \frac{(n-1)a_2}{2a_1},$$

which contradicts Lemma 7.2. Therefore $p'(z)$ must have a zero in $D((a+1)/a_2, a_1/a_2)$ and moreover in $D(a, \sigma_n(a))$, where $\sigma_n(a) = a + n(1-a)/a_2$. It is easy to see that $\sigma_n(a)$ attains its maximum on the unit interval at $a = a_n = (n+2-2\sqrt{n})/(n-2)$ and is decreasing on $[a_n, 1]$. This, with Theorem 7.2 shows that if r_n denotes the smallest positive number such that $D(a, r_n)$ contains at least one zero of $p'(z)$, then

$$r_n \leq \max_{a_n \leq a \leq 1} \min\{(1+a)^{1/n}, \sigma_n(a)\} =: \tau_n.$$

For $n \geq 9$, we have

$$\tau_n \leq 2^{1/n} \leq 2^{1/9} < 1.08006 \ . \qquad \square$$

From Corollary 7.1 and Aziz' Theorem it follows

Corollary 7.3 *For every $p \in \mathcal{P}$, the inequality*

$$h(A(p), A'(p)) < 1.08006$$

holds.

8 Special classes of polynomials

8.1 Polynomials with real critical points

In this section we consider problems from the Hausdorff geometry of polynomials with only real critical points and real coefficients. The problem to verify Conjecture 1 for polynomials with real coefficients and only real critical points was formulated in 1977 by G. Schmeisser [79, p. 173], See also Th. M. Rassias (1991) [72].

Conjecture 18 (Schmeisser, G (1977) [79]). *Conjecture 1 is true for all polynomials in $\mathcal{P}$ having real coefficients and only real critical points.*

J. E. Brown (1999) [26] considered such problems and noted for Conjecture 1 [26, p. 51]: "More surprisingly, it is unresolved even if p has real coefficients and real critical points".

Definition 8.1 *Set the notations:*

1. *$\mathcal{R}$ - the set of polynomials $p \in \mathcal{P}$ with only real critical points and real coefficients;*

2. *$\mathcal{R}_n$ - the set of polynomials $p \in \mathcal{R}$ of degree n.*

Lemma 8.1 *If $p \in \mathcal{R}_n$, then $p'(z) = n(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_{n-1})$, where*

$$-1 \le \zeta_1 \le \zeta_2 \le \cdots \le \zeta_{n-1} \le 1 \tag{8.32}$$

and $0 \in [\zeta_1, \zeta_{n-1}] = H(p')$.

Proof. The inequalities (8.32) follow from Gauss–Lucas Theorem 2.1. By Corollary 2.1 p has at least one zero $z_k = e^{i\varphi}$ with $-\pi/2 \le \varphi \le \pi/2$. From this and Lemma 5.1 there follows that p' has at least one zero in the disk $D(z_k/2, 1/2)$. As all zeros of p' are real, then $\zeta_{n-1} \ge 0$. In the same way we have $\zeta_1 \le 0$. $\square$

Corollary 8.1 *If $z_1 \in D(1,1) \bigcup D(-1,1)$ is a zero of p, then z_1 is a normal zero.*

Corollary 8.2 *Conjecture 3 and Conjecture 8 are true for every $p \in \mathcal{R}$.*

For the supremal polynomials in $\mathcal{R}$ for $\rho(A(p); A(p'))$, J. E. Brown proved in 1999 the following:

Theorem 8.1 (J. E. Brown [26]). *If p is supremal in $\mathcal{R}_n$ for $\rho(A(p); A(p'))$, $n \ge 4$, $p(z_1) = 0$, $|z_1| < 1$ and $z_1 \notin D(1,1) \bigcap D(-1,1)$, then*

$$\rho(z_1; A(p')) \le \left[\frac{2}{n}|\Im(z_1)|(1+|z_1|^2)(1+|z_1|)^{n-4}\right]^{\frac{1}{n-1}}.$$

Corollary 8.3 *If p is supremal in $\mathcal{R}_n$, $n \ge 4$, for $\rho(A(p); A(p'))$, $p(z_1) = 0$ and $|z_1| < \delta_n$, where δ_n is the smallest positive root of*

$$2x(1+x^2)(1+x)^{n-4} - n = 0,$$

then z_1 is a normal zero.

In J. E. Brown [26] other theorems for the supremal polynomials in $\mathcal{R}_n$ for $\rho(A(p); A(p'))$ are also proved, which are still not sufficient to prove Conjecture 1 for all polynomials $p \in \mathcal{R}$.

For the polynomials with real coefficients G. Schmeisser formulated

Conjecture 19 (G. Schmeisser (1977) [79]). *Conjecture 1 is true for all polynomials $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with real coefficients satisfying $0 \le a_0 \le a_1 \le \cdots \le a_n \le 1$.* (Note that automatically $p \in \mathcal{P}$ by the Enström-Kakeya Theorem).

8.2 Polynomials with real zeros

In this section we consider problems from the Hausdorff geometry of polynomials with only real zeros. This very special case is far from trivial and offers interesting problems.

Definition 8.2 *Set the notations:*

1. $\mathcal{S}$ *- the set of polynomials* $p \in \mathcal{P}$ *with only real zeros;*

2. $\mathcal{S}_n$ *- the set of polynomials* $p \in \mathcal{S}$ *of degree* n.

From Definition 8.2 it follows that every polynomial $p \in \mathcal{S}_n$ has the form

$$p(x) = (x - x_1)(x - x_2)\cdots(x - x_n) \quad \text{and} \quad p'(x) = n(x - \xi_1)(x - \xi_2)\cdots(x - \xi_{n-1}),$$

where

$$-1 = x_1 \le \xi_1 \le x_2 \le \xi_2 \le \cdots \le x_{n-1} \le \xi_{n-1} \le x_n = 1$$

and

$$H(p) = D_c(p) = [-1, 1].$$

For every $p \in \mathcal{S}_n$, we have:
$\rho(A(p'); A(p)) = \max\{\min\{x_{k+1} - \xi_k, \xi_k - x_k\} : \ k = 1, 2, \ldots, n-1\}$,
$\rho(A(p); H(p')) = \max\{\xi_1 + 1, 1 - \xi_{n-1}\}$,
$\rho(A(p); A(p')) = \max\{\rho(A(p); H(p')), \max\{\min\{x_k - \xi_{k-1}, \xi_k - x_k\} :$
$k = 2, 3, \ldots, n-1\}\}$,

Theorem 8.2 *Conjecture* 1^* *is true for every* $p \in \mathcal{S}$ *and it is strict for* $p \in \mathcal{S}_{2m}$.

Proof. From Lemma 5.1 there follows that for every $p \in \mathcal{S}_n$, the inequalities $\xi_1 \le 0$, $\xi_{n-1} \ge 0$ hold, therefore $\rho(A(p); A(p')) \le 1$. The inequality $\rho(A(p'); A(p)) \le 1$ is trivial, as $x_1 = -1$, $x_n = 1$. For sharpness sake, consider the polynomial $p = (x^2 - 1)^m \in \mathcal{S}_{2m}$, for which $\rho(A(p'); A(p)) = 1$.

8.2.1 Hausdorff deviations

From Theorem 8.2 $\overline{h}(A(\mathcal{S}); A(\mathcal{S}')) = 1$ follows.

Problem 5 *Find* $\underline{h}(A(\mathcal{S}); A(\mathcal{S}'))$.

To solve Problem 5, we have to find only $\underline{\rho}(A(\mathcal{S}'); A(\mathcal{S}))$ as $\underline{\rho}(A(\mathcal{S}); A(\mathcal{S}')) = 0$ by polynomials with only multiple zeros.

The estimate from above for $\rho(A(p); H(p'))$ in $\mathcal{S}_n$ is much smaller than in $\mathcal{P}_n$.

Theorem 8.3 *For every $p \in \mathcal{S}_n$, the inequality*

$$\rho(A(p); H(p')) \leq \frac{2}{n} \tag{8.33}$$

holds.

Proof. If $p \in \mathcal{S}_n$ and $p'(-1) = p'(1) = 0$, then $H(p') = [-1, 1]$ and $\rho(A(p); H(p')) = 0$. Let $p'(1) \neq 0$ (the case $p'(-1) \neq 0$ is considered in the same way) and let $p(x) = (x-1)q(x)$. Then

$$\frac{p''(1)}{p'(1)} = \sum_{j=1}^{n-1} \frac{1}{1-\xi_j} = 2\sum_{k=1}^{n-1} \frac{1}{1-x_k} = 2\frac{q'(1)}{q(1)}$$

or

$$\sum_{j=1}^{n-2} \frac{1}{1-\xi_j} + \frac{1}{1-\xi_{n-1}} = 1 + 2\sum_{k=2}^{n-1} \frac{1}{1-x_k}. \tag{8.34}$$

From (8.34), as $\xi_{k-1} \leq x_k$; $k = 2, 3, \ldots, n-1$, we have

$$\begin{aligned} \frac{1}{1-\xi_{n-1}} &= 1 + \sum_{k=2}^{n-1} \frac{1}{1-x_k} + \sum_{k=2}^{n-1} \left(\frac{1}{1-x_k} - \frac{1}{1-\xi_{k-1}} \right) \\ &\geq 1 + \sum_{k=2}^{n-1} \frac{1}{1-x_k} \geq 1 + \frac{n-2}{2} = \frac{n}{2} \end{aligned}$$

or

$$\xi_{n-1} \geq 1 - \frac{2}{n} \quad \text{and} \quad x_n - \xi_{n-1} = 1 - \xi_{n-1} \leq \frac{2}{n}.$$

In the same way we see that $\xi_1 - x_1 = \xi_1 + 1 \leq 2/n$, therefore

$$\rho(A(p); H(p')) \leq \frac{2}{n}. \qquad \square$$

Corollary 8.4 *The polynomials*

$$(x-1)(x+1)^{n-1} \quad \text{and} \quad (x+1)(x-1)^{n-1}$$

are supremal in $\mathcal{S}_n$ for $\rho(A(p); H(p'))$.

8.3 Polynomials with zeros in a half-disk

In 1996, A. Byrne [28] considered polynomials with zeros in a half-disk.

Definition 8.3 *Set the notations*

1. *I^+ - the set $\{z : \Im(z) \geq 0\}$;*

2. $\mathcal{T}$ - the set of polynomials $p \in \mathcal{P}$ with zeros in the half disk $D \bigcap I^+$;

3. $\mathcal{T}_n$ - the set of polynomials $p \in \mathcal{T}$ of degree n.

From the Definition 8.3 it follows

Lemma 8.2 *If $p \in \mathcal{T}$, then $p(-1) = p(1) = 0$.*

Theorem 8.4 *Conjecture 3 and Conjecture 8 are true for every $p \in \mathcal{T}$.*

Proof. It suffices to prove that for $p \in \mathcal{T}$,

$$\rho(H(p); H(p')) \leq 1, \tag{8.35}$$

as by Lemma 2.1

$$\rho(A(p); H(p')) \leq \rho(H(p); H(p')).$$

By Lemma 5.1, there are at least one zero ζ_1 of p' in the half disk $D(1/2, 1/2) \bigcap I^+$ and at least one zero ζ_2 of p' in the half disk $D(-1/2, 1/2) \bigcap I^+$. Then $H(p')$ contains a point $i\lambda$, with $0 \leq \lambda \leq 1/2$. As all zeros of p, which lie in

$$V = \left[\left(D \bigcap D(-1,1)\right) \bigcup \left(D \bigcap D(1,1)\right)\right] \bigcap I^+$$

are normal, we have to consider only the zeros in $W = (D \bigcap I^+) \backslash V$. It is geometrically clear that for every $z \in W$, the inequality $|z - i\lambda| \leq 1$ holds for $0 \leq \lambda \leq 1$. □

9 Related problems

In 1968 A. G. Goodman, Q. I. Rahman and J. Ratti [39] formulated a much general problem than Conjecture 1.

Problem 6 (Goodman, A. G., Q. I. Rahman and J. Ratti [39]). *Let $p \in \mathcal{P}_n(a)$; $0 \leq a \leq 1$. Let $\mathcal{D}(a, n)$ be a minimal region with the property that p' has at least one zero in $\mathcal{D}(a, n)$. Describe $\mathcal{D}(a, n)$.*

This problem was solved only for $n = 3$ by E. B. Saff and J. B. Twomey (1971) [76]. In J. Dronka and J Sokól (1999) [35] a similar problem is formulated.

Problem 7 (J. Dronka and J Sokól [35]). *Let $p \in \mathcal{P}(a)$ and $0 \leq a \leq 1$. Find a possibly large set, which is contained in $D(a, 1)$, such that there exists a polynomial $q \in \mathcal{P}(a)$ with no critical points in this set.*

In J. Dronka and J Sokól (1999) [35] it is proved that such set exists and it is not unique.

A very general problem is the following:

Problem 8 (Th. M. Rassias, see Milovanović, G. V., D. S. Mitrinović and Th. M. Rassias [64]). *Investigate the Gauss–Lucas Theorem and Conjecture 1 for polynomials with two and more variables.*

9.1 Zeros of higher derivatives

Up to now, we have considered only the relations between the zeros of a polynomial and the zeros of its first derivative. Here, some problems involving the zeros of the higher derivatives of a polynomial are considered.

A. Meir and A. Sharma (1969) [59] considered the deviation of the zeros of a special polynomial from the zeros of its s^{th} derivative.

Theorem 9.1 (A. Meir and A. Sharma [59]). *Let $p \in \mathcal{P}_n(1)$ and let the zero 1 be of multiplicity ν. Then at least one zero of $p^{(s)}(z)$, $1 \le s \le n-1$, lies in the disk $D(\nu/(s+1), 1-\nu/(s+1))$.*

We will use a Corollary of Theorem 9.1.

Corollary 9.1 *If $p \in \mathcal{P}_n(1)$, $s = 1, 2, \ldots, n-1$, then at least one zero of $p^{(s)}$ lies in the disk $D(1/(s+1), s/(s+1))$ and, moreover, in the disk $D(1, 2s/(s+1)) \supset D(1/(s+1), s/(s+1))$.*

Proof. Let $p(z) = (z-1)q(z)$, $\zeta_1, \zeta_2, \ldots, \zeta_{n-s}$ be the zeros of $p^{(s)}$ and $\eta_1, \eta_2, \ldots, \eta_{n-s}$ be the zeros of $q^{(s-1)}$. By Gauss–Lucas Theorem 2.1 $|\eta_k| \le 1$; $k = 1, 2, \ldots, n-s$. If $p^{(s)}(1) = 0$, we are done. Let $p^{(s)}(1) \ne 0$, then

$$\frac{p^{(s+1)}(1)}{p^{(s)}(1)} = \frac{s+1}{s} \cdot \frac{q^{(s)}(1)}{q^{(s-1)}(1)}$$

and

$$\sum_{j=1}^{n-s} \Re\left(\frac{1}{1-\zeta_j}\right) = \frac{s+1}{s} \sum_{k=1}^{n-s} \Re\left(\frac{1}{1-\eta_k}\right) \ge \frac{(s+1)(n-s)}{2s}.$$

If $\Re(1/(1-\zeta_1)) = \max\{\Re(1/(1-\zeta_j)) : \ j = 1, 2, \ldots, n-s\}$, then

$$\Re\left(\frac{1}{1-\zeta_1}\right) \ge \frac{s+1}{2s},$$

hence

$$\zeta_1 \in D(1/(s+1), s/(s+1)) \subset D(1, 2s/(s+1)). \qquad \square$$

9.1.1 Generalizations of statements for the first derivative

Lemma 9.1 *If $p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$ and $a_0 \ne 0$, then p has at least one zero in the disk $D(0, r)$, where $r = |a_m/a_0|^{1/m}$.*

Proof. The polynomial $q(z) = a_0 z^m - (-1)^m a_m$ is apolar to the polynomial p and all of the zeros of q lie in the disk $D(0, |a_m/a_0|^{1/m})$. The proof of the lemma follows from the Grace Apolarity Theorem 2.3. $\square$

We may generalize Corollary 5.1 in the following way.

Lemma 9.2 *If $p \in \mathcal{P}_n(a)$ and $|p^{(s)}(a)| \le (n!/(n-s)!)r^{n-s}$, $s = 1, 2, \ldots, n-1$, then at least one zero of $p^{(s)}(z)$ lies in the disk $D(a, r)$.*

Proof. We have

$$p^{(s)}(a-z) = (-1)^{n-s}\frac{n!}{(n-s)!}z^{n-s} + \cdots + p^{(s)}(a).$$

By Lemma 9.1

$$\left|\frac{a_{n-s}}{a_0}\right| = \left|\frac{p^{(s)}(a)(n-s)!}{n!}\right| = r^{n-s},$$

or $r = |a_m/a_0|^{1/(n-s)}$. $\square$

Lemma 9.3 *For every $p \in \mathcal{P}_n \bigcup \mathcal{G}_n$ and $s \in \{1, 2, \ldots, n-1\}$, the following inequality*

$$|p^{(s)}(z_1)| \le s!\binom{n-1}{n-s}(1+|z_1|)^{n-s}$$

holds.

Proof. From $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ and $|z_k| \le 1$; $k = 1, 2, \ldots, n$ there follows

$$|p^{(s)}(z_1)| \le s! \sum_{2 \le k_1 \le \cdots \le k_{n-s} \le n} |(z_1 - z_{k_1})(z_1 - z_{k_2})\cdots(z_1 - z_{k_{s-1}})|$$

$$\le s! \sum_{2 \le k_1 \le \cdots \le k_{n-s} \le n} (1+|z_1|)^{n-s} = s!\binom{n-1}{n-s}(1+|z_1|)^{n-s}. \quad \square$$

9.2 Generalization of Conjecture 1 for higher derivatives

9.2.1 Polynomials from $\mathcal{P}$

Conjecture 20 *If $p \in \mathcal{P}_n$ and $s = 1, 2, \ldots, n-1$, then*

$$\rho(A(p), A(p^{(s)})) \le \frac{2s}{s+1}.$$

For $s = 1$, Conjecture 20 is Conjecture 1.

Theorem 9.2 *Conjecture 20 is true for arbitrary n and $s = n-1$.*

Proof. Let $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$, then the only zero of $p^{(n-1)}(z)$ is $\zeta_1 = (z_1 + z_2 + \cdots + z_n)/n$ and

$$|z_1 - \zeta_1| = n^{-1}|(n-1)z_1 - (z_2 + z_3 + \cdots + z_n)|$$

$$\le n^{-1}(n-1+n-1) = \frac{2(n-1)}{n}. \quad \square$$

Corollary 9.2 *Conjecture 20 is true for all polynomials $p \in \mathcal{P}_3$.*

The case $s = 1$ is covered by Theorem 6.1 and the case $s = 2$, by Theorem 9.2.

Theorem 9.3 *Conjecture 20 is true for all polynomials $p \in \mathcal{P}_4$.*

Proof. By Theorem 6.2 and Theorem 9.2, it suffices to prove only the case $s = 2$ for polynomials $p \in \mathcal{P}_4(1)$.

1) For $|z_2| = |z_3| = |z_4| = 1$, the proof follows from Corollary 9.1.

2) For $|z_2| < 1$ and $|z_3| = |z_4| = 1$, without loss of generality, we may suppose that

$$z_2 = r_2 e^{i\varphi_1},\ z_3 = e^{i\varphi_3},\ z_4 = e^{i\varphi_4}, \quad \varphi_3 \le \varphi_2 \le \varphi_4 \quad \text{and} \quad \varphi_4 - \varphi_3 \le \pi.$$

In this situation

$$\left| z_2 - \frac{z_3 + z_4}{2} \right| \le 1 \quad \text{and} \quad |(z_2 - z_3)(z_2 - z_4)| \le 2. \tag{9.36}$$

Then, using (9.36), we have

$$|p''(z_2)| = 2 \left| 2(z_2 - 1)\left(z_2 - \frac{z_3 + z_4}{2}\right) + (z_2 - z_3)(z_2 - z_4) \right| \le 12$$

and by Lemma 9.2, at least one zero of p'' lies in the disk $D(z_2, 1)$. Then the proof follows from Corollary 9.1.

3) For $|z_2| < 1$, $|z_3| < 1$ and $|z_4| = 1$, we have first that $z_4 = -1$, as $p \in \mathcal{P}_4(1)$. Then

$$|p''(z_2)| = 2 \left| (z_2^2 - 1) + 2z_2(z_2 - z_3) \right| \le 12.$$

In the same way we get $|p''(z_3)| \le 12$. Then the proof follows from Corollary 9.1. $\square$

Conjecture 20 is sharp for $n = 3$ and for arbitrary n and $s = n - 1$. It is not sharp for $n = 4$ and $s = 2$. For the polynomial

$$p(z) = (z - 1)\left(z^2 + \frac{2}{3}z + 1\right) \quad \text{we have} \quad \rho(A(p); A(p'')) = \frac{2}{\sqrt{3}}.$$

Most probably $\overline{\rho}(A(\mathcal{P}_4); A(\mathcal{P}_4'')) = 2/\sqrt{3}$.

Theorem 9.4 *Conjecture 20 is true for a polynomial $p \in \mathcal{P}_n$ if*

$$|z_k| \le \left(\frac{2n}{s+1}\right)^{1/(n-s)} - 1 \quad \text{for} \quad k = 1, 2, \ldots, m$$

and $|z_k| = 1$ for $k = m, m+1, \ldots, n$.

Proof. The zeros $z_k;\ \ k = m, m+1, \dots, n$ are covered by Corollary 9.1. For the zeros $z_k;\ \ k = 1, 2, \dots, m$, from Lemma 9.3

$$|p^{(s)}(z_k)| \le s!\binom{n-1}{n-s}(1+|z_k|)^{n-s} \le s!\binom{n-1}{n-s}\frac{2n}{s+1} = \frac{n!}{(n-s)!}\cdot\frac{2s}{s+1}.$$

The proof follows from the last inequality and Lemma 9.2. $\square$

Remark 9.1 *For $s = 1$, from Theorem 9.4 it follows that a zero z_1 of a polynomial $p \in \mathcal{P}_n \bigcup \mathcal{G}_n$ is normal, if*

$$|z_1| \le n^{1/(n-1)} - 1. \tag{9.37}$$

From S. Vernon's Theorem 5.8 it follows that a zero z_1 of a polynomial $p \in \mathcal{P}_n \bigcup \mathcal{G}_n$ is normal, if

$$|z_1| \le \frac{1}{2}\sqrt{\frac{5}{4} - n^{1/(n+1)}}. \tag{9.38}$$

For $n \le 22$ the right part of equation (9.37) is larger than the right part of equation (9.38). For $n \ge 23$ the right part of equation (9.38) is larger than the right part of equation (9.37).

The following generalization of Aziz's Theorem 1.1 is proved as Theorem 3.3.

Theorem 9.5 *For every polynomial $p \in \mathcal{P}_n$ and $s = 1, 2, \dots, n-1$, the inequality $\rho(A(p^{(s)}); A(p)) \le 1$ holds.*

9.2.2 Hausdorff deviations

As an example, we have the following statement.

Theorem 9.6 *If $p \in \mathcal{P}_n$, then the inequalities*

$$\rho(A(p); H(p')) \le \rho(A(p); H(p'')) \le \dots \le \rho(A(p); H(p^{(n-1)}) = r_g(p)$$

and

$$\rho(H(p); H(p')) \le \rho(H(p); H(p'')) \le \dots \le \rho(H(p); H(p^{(n-1)})) = r_g(p)$$

hold.

Proof. By Gauss–Lucas Theorem 2.1, we have

$$c_g(p) = H(p^{(n-1)}) \subset H(p^{(n-2)}) \subset \dots \subset H(p') \subset H(p),$$

then the proof follows from Lemma 2.1. Observe that the only zero of $p^{(n-1)}(z)$ is the gravity center of p, see Definition 2.6, item 12. $\square$

In the same way we have

Theorem 9.7 *If $p \in \mathcal{G}_n$, then the inequalities*

$$\rho(A(p); H(p')) \le \rho(A(p); H(p'')) \le \cdots \le \rho(A(p); H(p^{(n-1)}) = 1$$

and

$$\rho(H(p); H(p')) \le \rho(H(p); H(p'')) \le \cdots \le \rho(H(p); H(p^{(n-1)})) = 1$$

hold.

Having in mind Theorem 9.6 and Theorem 9.7, one may expect that for every polynomial p of degree n and $s \in \{1, 2, \ldots, n-2\}$, the inequality

$$\rho(A(p); A(p^{(s)})) \le \rho(A(p); A(p^{(s+1)}))$$

holds.

9.2.3 Extremal polynomials

We have Phelps–Rodriguez Conjecture 4 that the only supremal polynomial in $\mathcal{P}_n(1)$ for $\rho(A(p); A(p'))$ is $p(z) = z^n - 1$.

The supremal polynomials in $\mathcal{P}_n(1)$ for $\rho(A(p); A(p^{(n-1)}))$ are two, $p(z) = (z-1)(z+1)^{n-1}$ and $(-1)^n p(-z)$.

Theorem 9.8

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(s)})) \ge \frac{2s}{n} \quad \text{for} \quad s > n/2$$

and

$$\overline{\rho}(A(\mathcal{P}_{2m}); A(\mathcal{P}_{2m}^{(m)})) \ge \frac{2\sqrt{m^2-1}}{2m-1}.$$

Proof. For the polynomial $p(z) = (z+1)^{n-1}(z-1)$ we have

$$p^{(s)}(z) = \frac{n!}{(n-s)!}(z+1)^{n-s-1}\left(z - 1 + \frac{2s}{n}\right)$$

and

$$\rho(A(p); A(p^{(s)})) = \frac{2s}{n}.$$

For the polynomial $p(z) = (z^2 + 2\alpha z + 1)(z-1)^{n-2}$ we have

$$p^{(s)}(z) = \frac{n!}{(n-s)!}(z-1)^{n-s-2}\left(z^2 + 2\frac{\alpha(n-s) - s}{n}z + 1 - \frac{2s(n-s)(1+\alpha)}{n(n-1)}\right).$$

For $s = n/2 = m$ and $\alpha = (2m-3)/(2m-1)$,

$$p^{(m)}(z) = \frac{(2m)!}{m!}(z-1)^{m-2}\left(z - \frac{1}{2m-1}\right)^2$$

and

$$\rho(A(p); A(p^{(m)})) = \left|\alpha + i\sqrt{1-\alpha^2} + \frac{1}{2m-1}\right| = \frac{2\sqrt{m^2-1}}{2m-1}. \qquad \square$$

Our last, very bold conjecture, is the following.

Conjecture 21

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(s)})) = 1 \quad \text{for} \quad s < n/2.$$

9.2.4 Polynomials with only real zeros

For the polynomials with only real zeros, we have the following generalization of Theorem 8.3.

Theorem 9.9 *For every $p \in \mathcal{S}_n$ and $s \in \{1, 2, \ldots, n-1\}$, the inequality*

$$\rho(A(p); H(p^{(s)})) \le \frac{2s}{n} \tag{9.39}$$

holds.

Proof. If $p \in \mathcal{S}_n$ and $p^{(s)}(-1) = p^{(s)}(1) = 0$, then $H(p(s))) = [-1, 1]$ and $\rho(A(p); H(p^{(s)})) = 0$. Let $p^{(s)}(1) \ne 0$ (the case $p^{(s)}(-1) \ne 0$ is considered in the same way). Let $p(x) = (x-1)q(x)$, $\xi_1, \xi_2, \ldots, \xi_{n-s}$ be the zeros of $p^{(s)}$, and $x_1, x_2, \ldots, x_{n-s}$ be the zeros of $q^{(s-1)}$. Then

$$-1 \le x_1 \le \xi_1 \le x_2 \le \xi_1 \le \cdots \le x_{n-s} \le \xi_{n-s} \le 1 \tag{9.40}$$

and

$$\frac{p^{(s+1)}(1)}{p^{(s)}(1)} = \sum_{j=1}^{n-s} \frac{1}{1-\xi_j} = \frac{s+1}{s} \sum_{k=1}^{n-s} \frac{1}{1-x_k} = \frac{s+1}{s} \frac{q^{(s)}(1)}{q^{(s-1)}(1)}$$

or

$$\sum_{j=1}^{n-s-1} \frac{1}{1-\xi_j} + \frac{1}{1-\xi_{n-s}} = \frac{1}{1-x_1} + \frac{s+1}{s} \sum_{k=2}^{n-s} \frac{1}{1-x_k}. \tag{9.41}$$

From (9.41), as $\xi_{k-1} \le x_k$; $k = 2, 3, \ldots, n-s$, we have

$$\frac{1}{1-\xi_{n-s}} = \frac{1}{1-x_1} + \frac{1}{s} \sum_{k=1}^{n-s} \frac{1}{1-x_k} + \sum_{k=2}^{n-s} \left(\frac{1}{1-x_k} - \frac{1}{1-\xi_{k-1}} \right)$$

$$\ge \frac{1}{2} + \frac{1}{s} \sum_{k=1}^{n-s} \frac{1}{1-x_k} \ge \frac{1}{2} + \frac{n-s}{2s} = \frac{n}{2s}$$

or

$$\xi_{n-s} \ge 1 - \frac{2s}{n} \quad \text{and} \quad 1 - \xi_{n-s} \le \frac{2s}{n}.$$

In the same way we see that $\xi_1 + 1 \le 2s/n$, therefore

$$\rho(A(p); H(p^{(s)})) \le \frac{2s}{n}. \qquad \square$$

Corollary 9.3 *The polynomials*

$$(x-1)(x+1)^{n-1} \quad \text{and} \quad (x+1)(x-1)^{n-1}$$

are supremal in $\mathcal{S}_n$ for $\rho(A(p); H(p^{(s)}))$.

10 Appendices

10.1 Calendar of the partial proofs of Conjecture 1

In this appendix we list all found proofs, arranged by dates, of Conjecture 1 for particular sets of polynomials.

1. $\mathcal{P}_3$.

   Brannan, D. A. (1968) [22]; Rubinstein, Z. (1968) [75]; Schmeisser, G. (1969) [77]; Meir, A. and A. Sharma (1969) [59]; Joyal, A. (1969) [42]; Saff, E. B. and J. B. Twomey (1971) [76]; Cohen, G. L. and G. H. Smith (1988) [31]; Borcea, J. (1996) [18], [19]; Todorov, P. G. (1996) [89].

2. $\mathcal{P}_{n,3}$.

   Saff, E. B. and J. B. Twomey (1971) [76].

3. $\mathcal{P}_4$.

   Rubinstein, Z. (1968) [75]; Schmeisser, G. (1969) [77]; Meir, A. and A. Sharma (1969) [59]; Joyal, A. (1969) [42]; Cohen, G. L. and G. H. Smith (1988) [30]; Borcea, J. (1996) [18], [19].

4. $\mathcal{P}_{n,4}$; $n \geq 4$.

   Brown, J. E. (1988) [23]; Kumar, S. and B. G. Shenoy (1990) [48].

5. $\mathcal{P}_5$.

   Meir, A. and A. Sharma (1969) [59]; Kumar, S. and B. G. Shenoy (1992) [50]; Borcea, J. (1996) [18], [19].

6. $\mathcal{P}_6$. Brown, J. (1991) [24]; Borcea, J. (1996) [18], [19].

7. $\mathcal{P}_{7,6}$.

   Brown, J. E. (1988) [23].

8. $\mathcal{P}_{n,6}$. Katsoprinakis, E. S. (1992) [45]; Katsoprinakis, E. S. (1996) [46].

9. $\mathcal{P}_7$.

   Borcea, J. (1996) [19]; Brown, J. (1997) [25];

10. $\mathcal{P}_{n,7}; n \geq 11$.

    Kumar, S. and B. G. Shenoy (1992) [50].

11. $\mathcal{P}_8$.

    Brown, J. and G. Xiang (1999) [27].

12. $\mathcal{P}_{n,8}$.

    Brown, J. and G. Xiang (1999) [27].

13. $\mathcal{P}_{n,m}$ if $n \geq 2^{m-1}$.

    Cohen, G. L. and G. H. Smith (1988).

14. $p \in \mathcal{P}$ with $H(A(p))$ a segment.

    Phelps, D. and R. S. Rodriguez (1972) [67]; Schmeisser, G. (1969) [77].

15. $p \in \mathcal{P}$ with $H(A(p))$ a triangle.

    Schmeisser, G (1977) [79].

16. All corners of $H(A(p))$ lie in the circumself circle of $A(p)$.

    Rubinstein, Z. (1968) [75]; Schmeisser, G. (1969) [77]; Goodman, A. G., Q. I. Rahman and J. Ratti (1969) [39].

17. $0 \in A(p)$.

    Schmeisser, G. (1969) [77]; Meir, A. and A. Sharma (1969) [59]; Gacs, G. (1971) [36].

18. $p \in \mathcal{P}_n$ with $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$, where a_k are real and $a_k \leq 0,\ k = 1, 2, \ldots, n$.

    Schmeisser, G (1971) [78].

19. $p \in \mathcal{P}_n$ with $p(z) = z^n + a_1 z^{n_1} + a_2 z^{n_2}$, where $n > n_1 > n_2$.

    Schmeisser, G (1971) [78].

20. $p \in \mathcal{P}_n$ with $p(z) = z^n + a_1 z^{n_1} + a_2 z^{n_2} + a_3 z^{n_3}$, where $n > n_1 > n_2 > n_3$.

    Schmeisser, G (1977) [79].

21. $p \in \mathcal{P}_n$ with $p(z) = \sum_{k=0}^{m} a_k z^{n_k}$ with $n = n_m \geq 3k - 2$, where $n = n_m > n_{m-1} > \cdots > n_0$.

    Schmeisser, G (1977) [79].

10.2 Autobiographical note

In the spring of 1956, I graduated from the Faculty of Physics and Mathematics of Sofia University (equivalent to a Masters degree), having already published several papers (in *Doklady AN USSR* and in some Bulgarian mathematical journals) in the field of *Real functions* and *Functional analysis*. For me it was natural to stay at the University as a Ph. D. student, or as an Assistant Professor, but because of some "peculiarities" in my biography, I became a mathematics teacher in a small village.

On April 1, 1958, after a petition to the Ministry of education, signed by all Professors of mathematics at Sofia University, I was allowed to become an assistant of Academician Nikola Obrechkoff. He was the most famous Bulgarian mathematician and a world known specialist on the distribution of the zeros of the algebraic polynomials. As an assistant of Acad. Obrechkoff, I wanted to do something, which would be of interest to him. After the acquaintance with some fundamental theorems for the distribution of the zeros and the critical points of the algebraic polynomials, I tried to invent something new in this field. In the middel of 1958 I formulated Conjecture 1, after experimenting with simple polynomials and capturing the polynomial $p(z) = z^n - 1$ as an extremal polynomial for the Grece–Heavood theorem [8, p. 149].

In 1959 I had a proof of Conjecture 1 for $n = 3$, but did not propose this proof for publication. I hesitated to bother Acad. Obrechkoff with such a small result, trying to prove Conjecture 1 for arbitrary n. Nevertheless, I acquainted many of my colleagues with Conjecture 1, including Professor L. Ilieff [12], a well known specialist in the Theory of the complex functions.

In the spring of 1959, Prof. L. Ilieff invited me to prepare a one year course of lectures on *Numerical Methods* for the academic year 1959/1960 in Sofia University. The idea was to start educating the mathematics students to solve mathematical problems with computers.

During the academic year 1960/1961, I specialized at Moscow State University (MGU) in *Numerical Methods* under the guidance of Acad. A. N. Tichonov. At MGU I attended lectures in Numerical methods, Programming of electronic computers, Linear programing and many others. My research interests were related to the seminar in MGU by Acad. A. N. Kolmogorov on ϵ-entropy, ϵ-capacity and widths of functional spaces and the seminar by Acad. S. M. Nikolskii at "Steklov Institut" on Approximation theory. In this situation, my attention was completely diverted from Conjecture 1.

I remembered Conjecture 1, when in August 1962, with Acad. Obrechkoff, Prof. L. Ilieff and several other Bulgarian mathematicians, I found myself at the International Congress of Mathematicians in Stockholm, supported as a young mathematician by the International Mathematics Union. I asked Acad. Obrechkoff to introduce me to Prof. M. Marden. The meeting with this extraordinary mathematician was short, but very helpful. I am grateful to him that he wrote in [56]: "*This conjecture was included in the collection of Research Problems in Function Theory, published in 1967 by Professor Hayman [40]. Since it had been brought to Hayman's attention by Professor Ilyeff, it became known as "Ilyeff's conjecture". Actually, Conjecture 1 was due to the Bulgarian mathematician Bl. Sendov who had aquainted me and probably others with it in 1962 at the International congress of Mathematics held in Stockholm.*"

Conjecture 1 became known as *Ilieff's conjecture* in the following way. L. Ilieff and W. K Hayman attended an International Conference on the *Theory of analytic functions* in Erevan, 6 - 13 Sept. 1965. At this conference, Prof. Ilieff formulated Conjecture 1, mentioning its author. Prof. Hayman remembered the conjecture as coming from Ilieff and included it in his book [40, Problem 4.5.] as *Ilieff's conjecture.*

After the publication of Hayman's book [40], Prof. Ilieff tried very hard to correct this mistake, and wrote several letters. I know very well, that this is not his fault and that it is not important.

More than 40 years after the formulation of Conjecture 1, I came back to it with the curiosity to study all the papers connected with it. There are more than 70 of these papers and half of them have been published during the last decade.

Acknowledgment. I would like to thank Professor Asen Dontchev for his generous help.

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Received April 18, 2001.