Absolute Stability Properties of the Richardson Extrapolation Combined with Explicit Runge-Kutta Methods – Extended Version

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1. Stability function of one-step numerical methods for solving systems of ODEs

Consider the classical initial value problem for non-linear systems of ordinary differential equations (ODEs):

$$\textbf{(1)} \qquad \mathbf{y}' = \mathbf{f}(\mathbf{t}, \mathbf{y}), \qquad \mathbf{t} \in [\mathbf{a}, \mathbf{b}], \qquad \mathbf{b} > \mathbf{a}, \qquad \mathbf{y} \in \mathbf{D} \subset \mathbf{R}^s, \qquad \mathbf{s} \geq 1, \qquad \mathbf{y}(\mathbf{a}) = \eta \in \mathbf{D}.$$

Assume that the initial value vector $\eta \in \mathbf{R}^s$ is given. The exact solution $\mathbf{y(t)}$ of the system defined by (1) is normally not known. Therefore, it is convenient to apply a suitable numerical method in order to calculate some sufficiently accurate approximate values of the components of the exact solution vector $\mathbf{y(t)}$ at the grid-points belonging to some discrete set of values of the time-variable. An example for such a set is given below:

$$\textbf{(2)} \qquad \mathbf{t}_0 = \mathbf{a}, \qquad \mathbf{t}_n = \mathbf{t}_{n-1} + \mathbf{h} = \mathbf{t}_0 + \mathbf{n}\mathbf{h} \quad (\mathbf{n} = 1, 2, \ldots, \mathbf{N}), \qquad \mathbf{t}_N = \mathbf{b}, \qquad \mathbf{h} = \frac{\mathbf{b} - \mathbf{a}}{\mathbf{N}}.$$

One of the basic requirements to the numerical methods that are suitable for the treatment of (1) can be explained in the following way. Assume that the exact solution $\mathbf{y(t)}$ of (1) is bounded, which is very often the case for problems that arise in different fields of science and engineering. Then it is desirable to establish the following property: the approximate solution, which is obtained by the selected numerical method at the grid-points of (2), must also be bounded. The natural requirement for obtaining a bounded numerical solution, in the case when the exact solution is bounded, leads, roughly speaking, to some stability requirements related to the numerical methods. Dahlquist [2] suggested in 1963 to study the stability properties of the selected numerical method by applying it in the solution of the scalar test-equation:

$$(3)\qquad y' = \lambda\, y, \quad t \in [0, \infty], \quad y \in C, \quad \lambda \in \alpha + \beta i, \quad \alpha \le 0, \quad y(0) = \eta .$$

The constant λ is assumed to be a given complex number with a non-positive real part and, therefore, in this particular case the dependent variable y takes values in the complex plane. Note that the initial value η is in general also a complex number.

The exact solution $y(t)$ of (3) is given by

$$(4)\qquad y(t) = \eta\, e^{\lambda t}, \qquad t \in [0, \infty].$$

It is clear that the exact solution $y(t)$ given by (4) is bounded when the constraint $\alpha \le 0$ that was introduced in (3) is satisfied. Therefore, it is necessary to require that the approximate solution computed by the selected numerical method is also bounded.

Assume now that (3) is treated by using some one-step numerical method for solving ODEs (the one-step methods are discussed, for example, in [1], [4]-[7], [10]). One can then successively compute approximations of function $y(t)$ from (4) at the grid-points of some grid, which is very similar to that defined by (2):

$$(5)\qquad t_0 = 0, \qquad t_n = t_{n-1} + h = t_0 + nh \quad (n = 1, 2, \ldots).$$

Also in this case the time-increment h is assumed to be some given positive constant.

The application of any one-step method in the numerical solution of (3) leads to the following recursive relation:

$$(6)\qquad y_n = R(\tilde{\lambda})\, y_{n-1} = \left(R(\tilde{\lambda})\right)^n y_0, \qquad \tilde{\lambda} = \lambda h, \qquad n = 1, 2, \ldots$$

The function $R(\tilde{\lambda})$ is called the stability function (see, for example, [7]). If the applied one-step method is explicit, then this function is a polynomial. It is a rational function (ratio of two polynomials) when implicit one-step methods are used. If the relation $|R(\tilde{\lambda})| \le 1$ is satisfied for some value of $\tilde{\lambda} = \lambda h$ then the selected one-step method will produce a bounded approximate solution of (3) for the applied value h of the time-stepsize. It is said that the numerical method is **absolutely stable** for this value of parameter $\tilde{\lambda}$ (see again [7]).

Consider the set of <u>**all**</u> parameters $\tilde{\lambda}$ for which the relationship $|R(\tilde{\lambda})| \le 1$ holds. This set is called <u>**absolute stability region**</u> of the numerical method under consideration ([7], p. 202).

The result for the scalar test-problem (3), which was sketched above, can easily be extended for linear systems of ODEs with constant coefficients that are written in the form:

$$\text{(7)} \qquad y' = Ay, \qquad t \in [0, \infty], \qquad y \in D \subset C^s, \qquad y(0) = \eta \in D.$$

It is assumed here that $A \in C^{s \times s}$ is a given constant and diagonalizable matrix and that η is some given vector. Under this assumption, there exists a non-singular matrix Q such that $Q^{-1}AQ = \Lambda$ where Λ is a diagonal matrix, whose diagonal elements are the eigenvalues of matrix A from (7). Substitute the expression $y = Q^{-1}z$ in (7). The result is:

$$\text{(8)} \qquad z' = \Lambda z, \qquad t \in [0, \infty], \qquad z \in \overline{D} \subset C^s, \qquad z(0) = \overline{\eta} = Q\eta \in \overline{D}.$$

It is clear that the system (8) consists of s independent scalar equations of type (3). Let $\lambda = \max(\lambda_1, \lambda_2, \ldots, \lambda_s)$ and $\tilde{\lambda} = \lambda h$. Assume also that the real parts of **all** eigenvalues are non-positive. Then the application of a one-step method in the solution of (8), and also of (7), will produce a bounded numerical solution when the inequality $|R(\tilde{\lambda})| \le 1$ satisfied.

Therefore, it is clear that for linear systems of ODEs with constant coefficients the absolute stability region is defined precisely in the same way as in the case where the scalar equation (3) is considered.

If matrix A is not constant, i.e. if $A = A(t)$ and, thus, if the elements of this matrix depend on the time-variable, then the above result is **no more** valid. Nevertheless, one can still expect stable results under certain assumptions. The main ideas can be sketched as follows. Assume that n is an arbitrary positive integer and that a matrix $A(\hat{t}_n)$ where $\hat{t}_n \in [t_{n-1}, t_n]$ is involved in the calculation of the approximation $y_n \approx y(t_n)$ by the selected numerical method. Assume further that matrix $A(\hat{t}_n)$ is diagonalizable. Then some diagonal matrix $\Lambda(\hat{t}_n)$ will appear in (8). Moreover, the eigenvalues of matrix $A(\hat{t}_n)$ will be diagonal elements of $\Lambda(\hat{t}_n)$. Let $\overline{\lambda}_n = \max\ (\lambda_1(\hat{t}_n), \lambda_2(\hat{t}_n), \ldots, \lambda_s(\hat{t}_n))$. Denote $\lambda = \max(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n)$ and $\tilde{\lambda} = \lambda h$. If the condition $|R(\tilde{\lambda})| \le 1$ is satisfied for any value of t_n belonging to (5), then one could expect the selected one-step method to be stable. However, it must again be noted that the stability is not guaranteed in this case.

Very similar considerations can also be applied for the non-linear system (1). In this case in stead of matrix $A(t)$ one should consider the Jacobean matrix $J(t)$ of the function $f(t,y)$ in the right-hand-side of (1).

The scalar equation (3) is very simple. This fact has been pointed out by many specialists in this field (see, for example, the remark on page 37 of [6]). The above considerations indicate that it is nevertheless worthwhile to base the absolute stability theory (at least until some more advanced and more reliable test-problem is found) on the simplest test-problem (3) as did Dahlquist [2] in 1963.

2. Stability polynomials of explicit Runge-Kutta methods

The general m-stage explicit Runge-Kutta method is a one-step numerical method for solving systems of ODEs. This numerical method is defined as follows (see more details in [7]):

$$\text{(9)} \qquad y_n = y_{n-1} + h \sum_{i=1}^{m} c_i k_i^n .$$

The coefficients c_i from (9) are some constants, while at time-step n the stages k_i^n are defined by

$$\text{(10)} \qquad k_1^n = f(t_{n-1}, y_{n-1}), \quad k_i^n = f\left(t_{n-1} + h a_i , y_{n-1} + h \sum_{j=1}^{i-1} b_{ij} k_j^n \right), \quad i = 2, 3, \ldots, m, \quad a_i = \sum_{j=1}^{i-1} b_{ij} ,$$

where b_{ij} are some constants depending of the particular numerical method.

Assume that the order of the explicit Runge-Kutta method is p and, additionally, that $p = m$ for the method under consideration. It can be shown (see [7]) that it is possible to satisfy $p = m$ only if $m \leq 4$ while we will necessarily have $p < m$ when m is greater than four. Assume further that the method defined with (9) and (10) is applied for the special problem (3). Then the stability polynomial $R(\tilde{\lambda})$ is given by (see [7], p. 202):

$$\text{(11)} \qquad R(\tilde{\lambda}) = 1 + \tilde{\lambda} + \frac{1}{2!}\tilde{\lambda}^2 + \ldots + \frac{1}{p!}\tilde{\lambda}^p , \qquad p = m, \qquad m = 1, 2, 3, 4.$$

Only **explicit** Runge-Kutta methods with $p = m$ will be considered in this paper.

3. Application of Richardson Extrapolation

Assume that the applied one-step numerical method is of order p and that an approximation of y_n has to be calculated under the assumption that a sufficiently accurate approximation y_{n-1} has already been computed. The classical Richardson Extrapolation (see [8]) can be performed in the following three steps under the assumption that the selected one-step numerical method is applied in the solution of (3):

Step 1	Perform a **<u>large</u>** step with a time-step $\mathbf{h}$ by using $\mathbf{y_{n-1}}$ as a starting value to calculate: **(12)** $\mathbf{z_n} = \mathbf{R}(\tilde{\lambda})\,\mathbf{y_{n-1}}$.
Step 2	Perform **<u>two small</u>** time-steps with a stepsize $\mathbf{0.5h}$ by using $\mathbf{y_{n-1}}$ as a starting value in the first of the two small time-steps: **(13)** $\hat{\mathbf{w}}_{\mathbf{n}} = \mathbf{R}\left(\frac{\tilde{\lambda}}{2}\right)\mathbf{y}_{\mathbf{n-1}}, \qquad \mathbf{w}_{\mathbf{n}} = \mathbf{R}\left(\frac{\tilde{\lambda}}{2}\right)\hat{\mathbf{w}}_{\mathbf{n}} = \left[\mathbf{R}\left(\frac{\tilde{\lambda}}{2}\right)\right]^2 \mathbf{y}_{\mathbf{n-1}}$.
Step 3	Compute (let us repeat here that $\mathbf{p}$ is the order of the selected numerical method) an improved solution by applying the Richardson Extrapolation: **(14)** $\mathbf{y}_{\mathbf{n}} = \frac{\mathbf{2^p\, w_n - z_n}}{\mathbf{2^p - 1}} = \frac{\mathbf{2^p}\left[\mathbf{R}\left(\frac{\tilde{\lambda}}{2}\right)\right]^2 - \mathbf{R}(\tilde{\lambda})}{\mathbf{2^p - 1}}\,\mathbf{y}_{\mathbf{n-1}}$.

The last relationship shows that the combination of the selected one-step numerical method and the Richardson Extrapolation can also be considered as a one-step numerical method when it is used to solve (3). It can easily be shown that the approximation $\mathbf{y_n}$ calculated by (14) is of order $\mathbf{p+1}$ and, therefore, it is more accurate than both $\mathbf{z_n}$ and $\mathbf{w_n}$ when the stepsize is sufficiently small (see, for example, [3] and [11]). Furthermore, it is clear that the stability polynomial of the combined numerical method is given by:

$$\textbf{(15)} \qquad \overline{\mathbf{R}}(\tilde{\lambda}) = \frac{\mathbf{2^p}\left[\mathbf{R}\left(\frac{\tilde{\lambda}}{2}\right)\right]^2 - \mathbf{R}(\tilde{\lambda})}{\mathbf{2^p - 1}} .$$

The above considerations are very general. They are valid for any one-step numerical method. However, as mentioned above, in the following part of this paper we shall restrict ourselves to the class of explicit Runge-Kutta methods with $\mathbf{p = m}$.

Our purpose will be to study **<u>the impact</u>** of the application of the Richardson Extrapolation on the stability properties of the underlying explicit Runge-Kutta methods. In other words, we shall compare the stability region of each of the explicit Runge-Kutta methods, for which $\mathbf{p = m}$ is satisfied, with the corresponding stability region obtained when the method under consideration is combined with the Richardson Extrapolation.

4. Impact of the Richardson Extrapolation on the absolute stability properties

The absolute stability regions of the explicit Runge-Kutta methods with $\mathbf{p=m}$ and $\mathbf{m=1, 2, 3, 4}$ are presented, for example, in [7], p. 202. In this section these stability regions will be compared with the absolute stability regions obtained when the Richardson Extrapolation is additionally used.

The boundaries of the absolute stability regions are obtained in the following way. Let $\tilde{\lambda}$ be equal to $\alpha + \beta i$ and ε be some small increment. Start with $\alpha = 0$ and test the values of the stability polynomial $R(\tilde{\lambda})$ for $\beta = 0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$. Continue this process as long as $R(\tilde{\lambda}) \le 1$ and denote by β_0 the last value for which the inequality $R(\tilde{\lambda}) \le 1$ was satisfied. Set $\alpha = -\varepsilon$ and repeat the same computations with $\beta = 0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$, to obtain the largest value β_ε for which $R(\tilde{\lambda}) \le 1$ is satisfied. Continuing in this way it will be possible to calculate the coordinates of a set of the points $(0, \beta_0)$, $(-\varepsilon, \beta_\varepsilon)$, $(-2\varepsilon, \beta_{2\varepsilon}), \ldots$ in the negative part of the complex plane. More precisely, all of these points are located close to the boundary of the part of the absolute stability region which is located over the real axis and to the left of the imaginary axis. Moreover, all these points lie inside the absolute stability region. Therefore, the curve connecting these points will be a close approximation of the boundary of the part of the stability region which is located over the real axis and to the left of the imaginary axis.

It should be mentioned here that $\varepsilon = 0.001$ was actually used in the preparation of all plots that are presented in this section.

It can easily be shown that the absolute stability region is symmetric with regard to the real axis. Therefore there is no need to repeat process that was sketched above for negative values of β .

Some people are drawing parts of the stability regions which are located to the right of the imaginary axis (see, for example, [7]). In our opinion this is not necessary and in the most of the cases it will not be desirable either. This can be explained as follows. Consider equation (3) and let again $\tilde{\lambda}$ be equal to $\alpha + \beta i$ but assume this time that α is positive. Then the exact solution (4) of (3) is not bounded and it is clearly not desirable to search for numerical methods which will produce bounded approximate solutions (the concept of relative stability, see [7], p. 75, is more appropriate in this situation, but this topic is beyond the scope of the present paper). Therefore, no attempts were made to find the parts of the stability regions which are located to the right of the imaginary axis.

The main advantages of the described in this section procedure for obtaining the absolute stability regions are two:

(a) it is conceptually very simple and

(b) it is very easy to prepare computer programs exploiting it.

The same procedure has also been used in [7]. Other procedures for drawing the absolute stability regions can be found in [4] - [7].

4.1. Combining the one-stage explicit Runge-Kutta method with the Richardson Extrapolation

The stability polynomial of the first-order one-stage explicit Runge-Kutta method (known also as the Forward Euler Formula or as the Explicit Euler Method) can be obtained from (11) by applying $\mathbf{p} = \mathbf{m} = \mathbf{1}$ in this formula:

$$\mathbf{(16)} \qquad \mathbf{R(\tilde{\lambda}) = 1 + \tilde{\lambda}} \ .$$

Denote by $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ the real and imaginary parts of the absolute stability polynomial $\mathbf{R(\tilde{\lambda})}$ in (16). If $\tilde{\lambda} = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{i}$ then it is obvious that

$$\mathbf{(17)} \qquad \overline{\mathbf{A}} = \mathbf{1} + \boldsymbol{\alpha}$$

and

$$\mathbf{(18)} \qquad \overline{\mathbf{B}} = \boldsymbol{\beta} \ .$$

The requirement $\mathbf{R(\tilde{\lambda})} \leq \mathbf{1}$ is now reduced to:

$$\mathbf{(19)} \qquad \sqrt{\overline{\mathbf{A}}^2 + \overline{\mathbf{B}}^2} \leq \mathbf{1}.$$

The application of the Richardson Extrapolation together with the first-order one-stage explicit Runge-Kutta method leads according to (15) for $\mathbf{p = 1}$ and (16), applied with arguments $\mathbf{0.5\tilde{\lambda}}$ and $\tilde{\lambda}$ respectively, to a stability polynomial of the form:

$$\mathbf{(20)} \qquad \overline{\mathbf{R}}(\tilde{\lambda}) = \mathbf{2}\left[\mathbf{1} + \frac{\tilde{\lambda}}{\mathbf{2}}\right]^2 - (\mathbf{1} + \tilde{\lambda}) \ .$$

This polynomial can be reduced to:

$$\mathbf{(21)} \qquad \overline{\mathbf{R}}(\tilde{\lambda}) = \mathbf{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{\mathbf{2}}.$$

The following values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ can be obtained by setting $\tilde{\lambda} = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{i}$ in (21):

(22) $$\overline{\mathbf{A}} = 1 + \alpha + \frac{\alpha^2}{2} - \frac{\beta^2}{2}$$

and

(23) $$\overline{\mathbf{B}} = \beta + \alpha\beta \, .$$

The requirement $\mathbf{R}(\tilde{\lambda}) \leq 1$ will now be satisfied when (19) holds with the values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ from (22) and (23).

The stability regions obtained by using (17) and (19) are given in Fig.1.

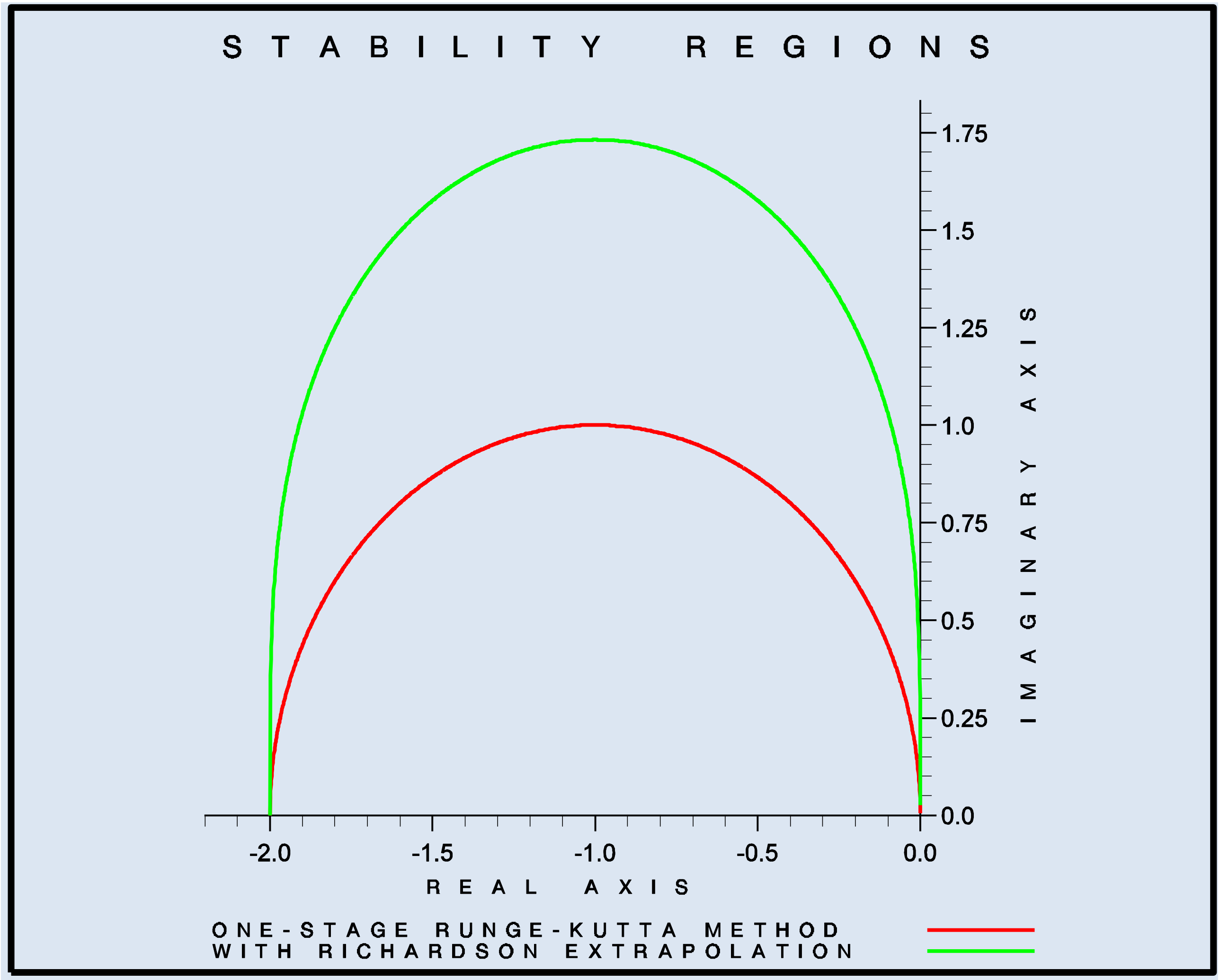

Figure 1

Stability regions of the original first-order one-stage explicit method and the combination of the Richardson Extrapolation with this method.

4.2. Combining the two-stage explicit Runge-Kutta method with the Richardson Extrapolation

The stability polynomial of the second-order two-stage explicit Runge-Kutta method can be obtained from (11) by setting $\mathbf{p = m = 2}$:

$$\textbf{(24)} \qquad \mathbf{R}(\tilde{\lambda}) = 1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} \; .$$

This is the same polynomial as that for the first-order one-stage explicit Runge-Kutta method combined with the Richardson Extrapolation: compare (24) with (21). Therefore, the values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ can also in this case be calculated by using (22) and (23) and the requirement $\mathbf{R}(\tilde{\lambda}) \le 1$ will now be satisfied when (19) holds with these values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$.

The application of the Richardson Extrapolation together with the second-order two-stage explicit Runge-Kutta method leads according to (15) for $\mathbf{p = 2}$ and (24), applied with arguments $\mathbf{0.5}\tilde{\lambda}$ and $\tilde{\lambda}$ respectively, to a stability polynomial of the form:

$$\textbf{(25)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{4}{3}\left[1 + \frac{\tilde{\lambda}}{2} + \frac{1}{2}\left(\frac{\tilde{\lambda}}{2}\right)^2\right]^2 - \frac{1}{3}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2}\right) \; .$$

The equality given below in (29) can easily be obtained from (25) by performing the transformations shown in (26) – (28):

$$\textbf{(26)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{4}{3}\left(1 + \frac{\tilde{\lambda}}{2} + \frac{\tilde{\lambda}^2}{8}\right)^2 - \frac{1}{3}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2}\right) ,$$

$$\textbf{(27)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{4}{3}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{4} + \frac{\tilde{\lambda}^2}{4} + \frac{\tilde{\lambda}^3}{8} + \frac{\lambda^4}{64}\right) - \frac{1}{3}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2}\right) ,$$

$$\textbf{(28)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{4}{3} + \frac{4}{3}\tilde{\lambda} + \frac{2}{3}\tilde{\lambda}^2 + \frac{1}{6}\tilde{\lambda}^3 + \frac{1}{48}\tilde{\lambda}^4 - \frac{1}{3} - \frac{1}{3}\tilde{\lambda} - \frac{1}{6}\tilde{\lambda}^2 ,$$

$$\textbf{(29)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = 1 + \tilde{\lambda} + \frac{1}{2}\tilde{\lambda}^2 + \frac{1}{6}\tilde{\lambda}^3 + \frac{1}{48}\tilde{\lambda}^4 \; .$$

Set $\tilde{\lambda} = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{i}$ in (29) and perform successively the transformations shown below:

$$(30) \qquad \overline{\mathbf{R}}(\alpha+\beta\mathbf{i}) = 1+\alpha+\beta\mathbf{i}+\frac{\alpha^2}{2}+\alpha\beta\,\mathbf{i}-\frac{\beta^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^2\beta\,\mathbf{i}}{2}-\frac{\alpha\beta^2}{2}-\frac{\beta^3\,\mathbf{i}}{6}$$

$$+\frac{\alpha^4}{48}+\frac{\alpha^3\beta\,\mathbf{i}}{12}-\frac{\alpha^2\beta^2}{8}-\frac{\alpha\beta^3\mathbf{i}}{12}+\frac{\alpha^4}{48}\ ,$$

$$(31) \qquad \overline{\mathbf{R}}(\alpha+\beta\mathbf{i}) = \left(1+\alpha+\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\frac{\alpha^3}{6}-\frac{\alpha\beta^2}{2}+\frac{\alpha^4}{48}-\frac{\alpha^2\beta^2}{8}+\frac{\beta^4}{48}\right)$$

$$+\left(\beta+\alpha\beta+\frac{\alpha^2\beta}{2}-\frac{\beta^3}{6}+\frac{\alpha^3\beta}{12}-\frac{\alpha\beta^3}{12}\right)\mathbf{i}\ .$$

It is clearly seen from (31) that the values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are in this case given by:

$$(32) \qquad \overline{\mathbf{A}} = 1+\alpha+\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\frac{\alpha^3}{6}-\frac{\alpha\beta^2}{2}+\frac{\alpha^4}{48}-\frac{\alpha^2\beta^2}{8}+\frac{\beta^4}{48}\ ,$$

$$(33) \qquad \overline{\mathbf{B}} = \beta+\alpha\beta+\frac{\alpha^2\beta}{2}-\frac{\beta^3}{6}+\frac{\alpha^3\beta}{12}-\frac{\alpha\beta^3}{12}\ .$$

The requirement $\mathbf{R}(\tilde{\lambda}) \leq 1$ will be satisfied when (19) holds with the values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ from (32) and (33).

The stability regions obtained by using the formulae derived in this sub-section are given in Fig.2.

4.3. Combining the three-stage explicit Runge-Kutta method with the Richardson Extrapolation

The stability polynomial of the third-order three-stage explicit Runge-Kutta method can be obtained from (11) by setting $\mathbf{p}=\mathbf{m}=3$

$$(34) \qquad \mathbf{R}(\tilde{\lambda}) = 1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{2}+\frac{\tilde{\lambda}^3}{6}\ .$$

By setting $\tilde{\lambda} = \alpha+\beta\mathbf{i}$ it can easily be established that the values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are given by:

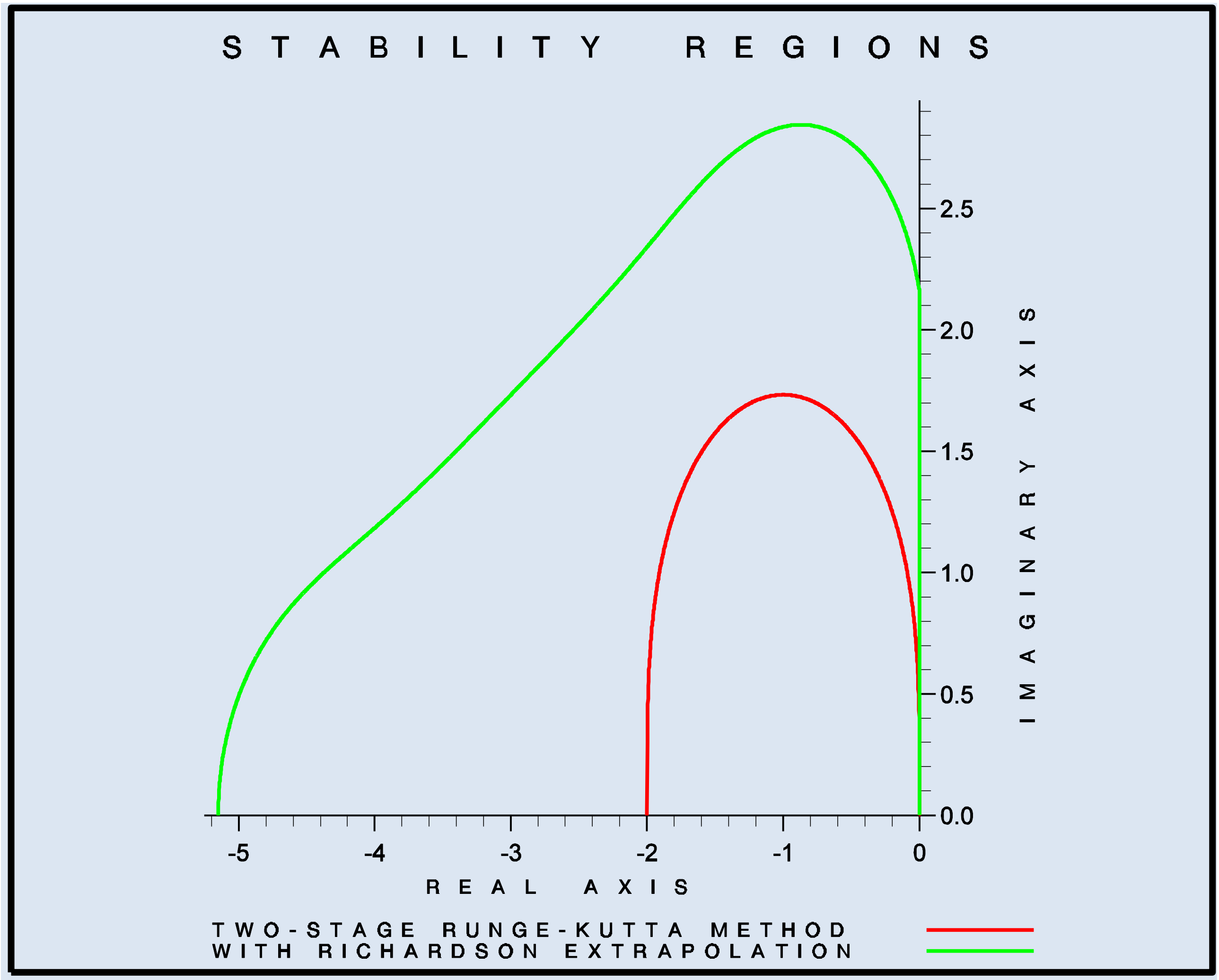

Figure 2

Stability regions of the original second-order two-stage explicit method and the combination of the Richardson Extrapolation with this method.

$$\textbf{(35)} \qquad \bar{A} = 1 + \alpha + \frac{\alpha^2}{2} - \frac{\beta^2}{2} + \frac{\alpha^3}{6} - \frac{\alpha\beta^2}{2}$$

and

$$\textbf{(36)} \qquad \bar{B} = \alpha + \alpha\beta + \frac{\alpha^2\beta}{2} - \frac{\beta^3}{6} \ .$$

The requirement $\mathbf{R}(\tilde{\lambda}) \leq \mathbf{1}$ will be satisfied when (19) holds with the values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ from (35) and (36).

The application of the Richardson Extrapolation together with the third-order three-stage explicit Runge-Kutta method leads according to (16) applied with $\mathbf{p} = \mathbf{3}$ to the following series of equalities:

$$\textbf{(37)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{\mathbf{8}}{\mathbf{7}}\left[\mathbf{R}\left(\frac{\tilde{\lambda}}{\mathbf{2}}\right)\right]^2 - \frac{\mathbf{1}}{\mathbf{7}}\mathbf{R}\left(\tilde{\lambda}\right),$$

$$\textbf{(38)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{\mathbf{8}}{\mathbf{7}}\left[\mathbf{1} + \frac{\tilde{\lambda}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\tilde{\lambda}}{\mathbf{2}}\right)^2 + \frac{\mathbf{1}}{\mathbf{6}}\left(\frac{\tilde{\lambda}}{\mathbf{2}}\right)^3\right]^2 - \frac{\mathbf{1}}{\mathbf{7}}\left(\mathbf{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{\mathbf{2}} + \frac{\tilde{\lambda}^3}{\mathbf{6}}\right),$$

$$\textbf{(39)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{\mathbf{8}}{\mathbf{7}}\left(\mathbf{1} + \frac{\tilde{\lambda}}{\mathbf{2}} + \frac{\tilde{\lambda}^2}{\mathbf{8}} + \frac{\tilde{\lambda}^3}{\mathbf{48}}\right)^2 - \frac{\mathbf{1}}{\mathbf{7}}\left(\mathbf{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{\mathbf{2}} + \frac{\tilde{\lambda}^3}{\mathbf{6}}\right),$$

$$\textbf{(40)} \qquad \bar{\mathbf{R}}(\tilde{\lambda}) = \frac{\mathbf{8}}{\mathbf{7}}\left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 - \frac{\mathbf{1}}{\mathbf{7}}\left(\mathbf{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{\mathbf{2}} + \frac{\tilde{\lambda}^3}{\mathbf{6}}\right).$$

The constants $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are defined by

$$\textbf{(41)} \qquad \hat{\mathbf{A}} = \mathbf{1} + \frac{\tilde{\lambda}}{\mathbf{2}} + \frac{\tilde{\lambda}^2}{\mathbf{8}},$$

$$\textbf{(42)} \qquad \hat{\mathbf{B}} = \frac{\tilde{\lambda}^3}{\mathbf{48}}.$$

Note that $\hat{\mathbf{A}}$ is the stability polynomial of the numerical method in the previous sub-section, the second-order two-stage explicit Runge-Kutta method.

Now it is clear that

$$\textbf{(43)} \qquad \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 = \hat{\mathbf{A}}^2 + \mathbf{2}\,\hat{\mathbf{A}}\,\hat{\mathbf{B}} + \hat{\mathbf{B}}^2 = \left(\mathbf{1} + \frac{\tilde{\lambda}}{\mathbf{2}} + \frac{\tilde{\lambda}^2}{\mathbf{8}}\right)^2 + \mathbf{2}\left(\mathbf{1} + \frac{\tilde{\lambda}}{\mathbf{2}} + \frac{\tilde{\lambda}^2}{\mathbf{8}}\right)\frac{\tilde{\lambda}^3}{\mathbf{48}} + \left(\frac{\tilde{\lambda}^3}{\mathbf{48}}\right)^2.$$

The first term in the right-hand-side of (43) has been calculated in the previous sub-section; see (27). By using this result the following relationships can be obtained:

(44) $$\left(\hat{A}+\hat{B}\right)^2 = 1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{4}+\frac{\tilde{\lambda}^2}{4}+\frac{\tilde{\lambda}^3}{8}+\frac{\lambda^4}{64}+\left(2+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{4}\right)\frac{\tilde{\lambda}^3}{48}+\frac{\tilde{\lambda}^6}{2304},$$

(45) $$\left(\hat{A}+\hat{B}\right)^2 = 1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{2}+\frac{\tilde{\lambda}^3}{8}+\frac{\lambda^4}{64}+\frac{\lambda^3}{24}+\frac{\tilde{\lambda}^4}{48}+\frac{\tilde{\lambda}^5}{192}+\frac{\tilde{\lambda}^6}{2304},$$

(46) $$\left(\hat{A}+\hat{B}\right)^2 = 1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{2}+\frac{\tilde{\lambda}^3}{6}+\frac{7\tilde{\lambda}^4}{192}+\frac{\tilde{\lambda}^5}{192}+\frac{\tilde{\lambda}^6}{2304}.$$

Inserting the right-hand-size of (46) in (40) the following expressions for the stability polynomial can be found:

(47) $$\overline{R}(\tilde{\lambda}) = \frac{8}{7}\left(1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{2}+\frac{\tilde{\lambda}^3}{6}+\frac{7\tilde{\lambda}^4}{192}+\frac{\tilde{\lambda}^5}{192}+\frac{\tilde{\lambda}^6}{2304}\right)-\frac{1}{7}\left(1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{2}+\frac{\tilde{\lambda}^3}{6}\right),$$

(48) $$\overline{R}(\tilde{\lambda}) = \frac{8}{7}+\frac{8\tilde{\lambda}}{7}+\frac{4\tilde{\lambda}^2}{7}+\frac{4\tilde{\lambda}^3}{21}+\frac{\tilde{\lambda}^4}{24}+\frac{\tilde{\lambda}^5}{168}+\frac{\tilde{\lambda}^6}{2016}-\frac{1}{7}-\frac{\tilde{\lambda}}{7}-\frac{\tilde{\lambda}^2}{14}-\frac{\tilde{\lambda}^3}{42},$$

(49) $$\overline{R}(\tilde{\lambda}) = 1+\tilde{\lambda}+\frac{\tilde{\lambda}^2}{2}+\frac{\lambda^3}{6}+\frac{\tilde{\lambda}^4}{24}+\frac{\tilde{\lambda}^5}{168}+\frac{\tilde{\lambda}^6}{2016}.$$

Set $\tilde{\lambda} = \alpha + \beta i$ in (49):

(50) $$\overline{R}(\alpha+\beta i) = 1+\alpha+\beta i+\frac{\alpha^2}{2}+\alpha\beta i-\frac{\beta^2}{2}+\frac{\alpha^3}{6}+\frac{\alpha^2\beta i}{2}-\frac{\alpha\beta^2}{2}-\frac{\beta^3 i}{6}$$

$$+\frac{\alpha^4}{24}+\frac{\alpha^3\beta i}{6}-\frac{\alpha^2\beta^2}{4}-\frac{\alpha\beta^3 i}{6}+\frac{\beta^4}{24}$$

$$+\frac{\alpha^5}{168}+\frac{5\alpha^4\beta i}{168}-\frac{10\alpha^3\beta^2}{168}-\frac{10\alpha^2\beta^3 i}{168}+\frac{5\alpha\beta^4}{168}+\frac{\beta^5 i}{168}$$

$$+\frac{\alpha^6}{2016}+\frac{6\alpha^5\beta i}{2016}-\frac{15\alpha^4\beta^2}{2016}-\frac{20\alpha^3\beta^3 i}{2016}+\frac{15\alpha^2\beta^4}{2016}+\frac{6\alpha\beta^5 i}{2016}-\frac{\beta^6}{2016}.$$

$$(51)\qquad \bar{\mathbf{R}}(\alpha+\beta i) = (1+\alpha+\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\frac{\alpha^3}{6}-\frac{\alpha\beta^2}{2}+\frac{\alpha^4}{24}-\frac{\alpha^2\beta^2}{4}+\frac{\beta^4}{24}$$

$$+\frac{\alpha^5}{168}-\frac{10\alpha^3\beta^2}{168}+\frac{5\alpha\beta^4}{168}+\frac{\alpha^6}{2016}-\frac{15\alpha^4\beta^2}{2016}+\frac{15\alpha^2\beta^4}{2016}-\frac{\beta^6}{2016})$$

$$+(\beta+\alpha\beta+\frac{\alpha^2\beta}{2}-\frac{\beta^3}{6}+\frac{\alpha^3\beta}{6}-\frac{\alpha\beta^3}{6}$$

$$+\frac{5\alpha^4\beta}{168}-\frac{10\alpha^2\beta^3}{168}+\frac{\beta^5}{168}+\frac{6\alpha^5\beta}{2016}-\frac{20\alpha^3\beta^3}{2016}+\frac{6\alpha\beta^5}{2016})i\ .$$

It is clear now that the values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are given by:

$$(52)\qquad \bar{\mathbf{A}} = 1+\alpha+\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\frac{\alpha^3}{6}-\frac{\alpha\beta^2}{2}+\frac{\alpha^4}{24}-\frac{\alpha^2\beta^2}{4}+\frac{\beta^4}{24}$$

$$+\frac{\alpha^5}{168}-\frac{10\alpha^3\beta^2}{168}+\frac{5\alpha\beta^4}{168}+\frac{\alpha^6}{2016}-\frac{15\alpha^4\beta^2}{2016}+\frac{15\alpha^2\beta^4}{2016}-\frac{\beta^6}{2016}$$

and

$$(53)\qquad \bar{\mathbf{B}} = \beta+\alpha\beta+\frac{\alpha^2\beta}{2}-\frac{\beta^3}{6}+\frac{\alpha^3\beta}{6}-\frac{\alpha\beta^3}{6}$$

$$+\frac{5\alpha^4\beta}{168}-\frac{10\alpha^2\beta^3}{168}+\frac{\beta^5}{168}+\frac{6\alpha^5\beta}{2016}-\frac{20\alpha^3\beta^3}{2016}+\frac{6\alpha\beta^5}{2016}\ .$$

The requirement $\mathbf{R}(\tilde{\lambda}) \leq 1$ will be satisfied when (19) holds with the values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ from (52) and (53).

The stability regions obtained by using the formulae derived in this sub-section are given in Fig.3.

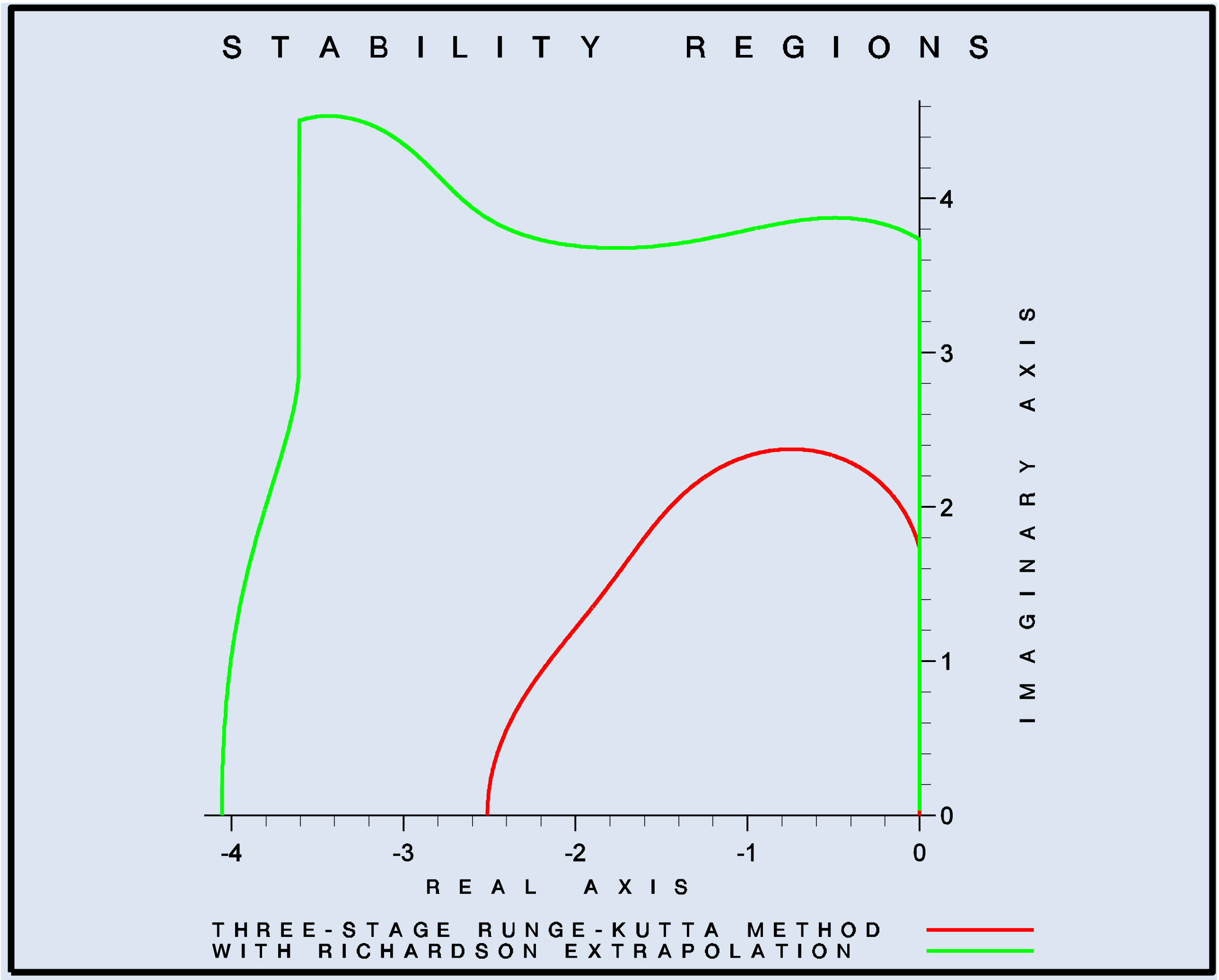

Figure 3

Stability regions of the original third-order three-stage explicit method and the combination of the Richardson Extrapolation with this method.

4.4. Combining the four-stage explicit Runge-Kutta method with the Richardson Extrapolation

The stability polynomial of the fourth-order four-stage explicit Runge-Kutta method can be obtained from (11) by setting $\mathbf{p}=\mathbf{m}=\mathbf{4}$

$$\textbf{(54)} \qquad \mathbf{R}(\tilde{\lambda}) = 1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24} .$$

The following values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ from can found by setting $\tilde{\lambda} = \alpha + \beta \mathbf{i}$ in (54):

(55) $$\bar{A} = 1 + \alpha + \frac{\alpha^2}{2} - \frac{\beta^2}{2} + \frac{\alpha^3}{6} - \frac{\alpha\beta^2}{2} + \frac{\alpha^4}{24} - \frac{\alpha^2\beta^2}{4} + \frac{\beta^4}{24}$$

and

(56) $$\bar{B} = \beta + \alpha\beta + \frac{\alpha^2\beta}{2} - \frac{\beta^3}{6} + \frac{\alpha^3\beta}{6} - \frac{\alpha\beta^3}{6} \; .$$

The requirement $R(\tilde{\lambda}) \le 1$ will be satisfied when (19) holds with the values of $\bar{A}$ and $\bar{B}$ from (55) and (56).

It is much more difficult to derive the condition for achieving absolute stability in the case where the fourth-order four-stage explicit Runge-Kutta method is combined with the Richardson Extrapolation.

The application of the Richardson Extrapolation together with the fourth-order four-stage explicit Runge-Kutta method leads according to (16) applied with $p = 4$ to the following series of equalities:

(57) $$\bar{R}(\tilde{\lambda}) = \frac{16}{15}\left[R\left(\frac{\tilde{\lambda}}{2}\right)\right]^2 - \frac{1}{15} R\left(\tilde{\lambda}\right),$$

(58) $$\bar{R}(\tilde{\lambda}) = \frac{16}{15}\left[1 + \frac{\tilde{\lambda}}{2} + \frac{1}{2}\left(\frac{\tilde{\lambda}}{2}\right)^2 + \frac{1}{6}\left(\frac{\tilde{\lambda}}{2}\right)^3 + \frac{1}{24}\left(\frac{\tilde{\lambda}}{2}\right)^4\right]^2 - \frac{1}{15}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24}\right),$$

(59) $$\bar{R}(\tilde{\lambda}) = \frac{16}{15}\left[1 + \frac{\tilde{\lambda}}{2} + \frac{\tilde{\lambda}^2}{8} + \frac{\tilde{\lambda}^3}{48} + \frac{\tilde{\lambda}^4}{384}\right]^2 - \frac{1}{15}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24}\right),$$

(60) $$\bar{R}(\tilde{\lambda}) = \frac{16}{15}\left(\hat{A} + \hat{B}\right)^2 - \frac{1}{15}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24}\right).$$

In this case the constants $\hat{A}$ and $\hat{B}$ are defined by

(61) $$\hat{A} = 1 + \frac{\tilde{\lambda}}{2} + \frac{\tilde{\lambda}^2}{8} + \frac{\tilde{\lambda}^3}{48}$$

and

$$\text{(62)} \qquad \hat{\mathbf{B}} = \frac{\tilde{\lambda}^4}{384}.$$

Note that $\hat{\mathbf{A}}$ is the stability polynomial of the numerical method in the previous sub-section, the third-order three-stage explicit Runge-Kutta method.

Now it is clear that the following relationships hold:

$$\text{(63)} \qquad \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 = \hat{\mathbf{A}}^2 + 2\hat{\mathbf{A}}\,\hat{\mathbf{B}} + \hat{\mathbf{B}}^2,$$

$$\text{(64)} \qquad \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 = \left(1 + \frac{\tilde{\lambda}}{2} + \frac{\tilde{\lambda}^2}{8} + \frac{\tilde{\lambda}^3}{48}\right)^2 + 2\left(1 + \frac{\tilde{\lambda}}{2} + \frac{\tilde{\lambda}^2}{8} + \frac{\tilde{\lambda}^3}{48}\right)\frac{\tilde{\lambda}^4}{384} + \left(\frac{\tilde{\lambda}^4}{384}\right)^2.$$

The first term in the right-hand-side of (64) has been calculated in the previous sub-section; see (46). By using this result the following equalities can be obtained:

$$\text{(65)} \qquad \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 = 1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{7\tilde{\lambda}^4}{192} + \frac{\tilde{\lambda}^5}{192} + \frac{\tilde{\lambda}^6}{2304} + \left(2 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{4} + \frac{\tilde{\lambda}^3}{24}\right)\frac{\tilde{\lambda}^4}{384} + \frac{\tilde{\lambda}^8}{147456},$$

$$\text{(66)} \qquad \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 = 1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{7\tilde{\lambda}^4}{192} + \frac{\tilde{\lambda}^5}{192} + \frac{\tilde{\lambda}^6}{2304} + \frac{\tilde{\lambda}^4}{192} + \frac{\tilde{\lambda}^5}{384} + \frac{\tilde{\lambda}^6}{1536} + \frac{\tilde{\lambda}^7}{9216} + \frac{\tilde{\lambda}^8}{147456},$$

$$\text{(67)} \qquad \left(\hat{\mathbf{A}} + \hat{\mathbf{B}}\right)^2 = 1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24} + \frac{\tilde{\lambda}^5}{128} + \frac{5\tilde{\lambda}^6}{4608} + \frac{\tilde{\lambda}^7}{9216} + \frac{\tilde{\lambda}^8}{147456}.$$

The following expressions for the stability polynomial can be found by inserting the right-hand-size of (67) in (60):

$$\text{(68)} \qquad \overline{\mathbf{R}}(\tilde{\lambda}) = \frac{16}{15}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24} + \frac{\tilde{\lambda}^5}{128} + \frac{5\tilde{\lambda}^6}{4608} + \frac{\tilde{\lambda}^7}{9216} + \frac{\tilde{\lambda}^8}{147456}\right)$$

$$- \frac{1}{15}\left(1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{6} + \frac{\tilde{\lambda}^4}{24}\right),$$

(69) $$\overline{\mathbf{R}}(\tilde{\lambda}) = \frac{16}{15} + \frac{16\tilde{\lambda}}{15} + \frac{8\tilde{\lambda}^2}{15} + \frac{16\tilde{\lambda}^3}{120} + \frac{2\tilde{\lambda}^4}{45} + \frac{\tilde{\lambda}^5}{120} + \frac{\tilde{\lambda}^6}{864} + \frac{\tilde{\lambda}^7}{8640} + \frac{\tilde{\lambda}^8}{138240}$$

$$-\frac{1}{15} - \frac{\tilde{\lambda}}{15} - \frac{\tilde{\lambda}^2}{30} - \frac{\tilde{\lambda}^3}{120} - \frac{\tilde{\lambda}^4}{360} \ ,$$

(70) $$\overline{\mathbf{R}}(\tilde{\lambda}) = 1 + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2} + \frac{\tilde{\lambda}^3}{8} + \frac{\tilde{\lambda}^4}{24} + \frac{\tilde{\lambda}^5}{120} + \frac{\tilde{\lambda}^6}{864} + \frac{\tilde{\lambda}^7}{8640} + \frac{\tilde{\lambda}^8}{138240} \ .$$

Set $\tilde{\lambda} = \alpha + \beta \mathbf{i}$ in (70):

(71) $$\overline{\mathbf{R}}(\alpha + \beta \mathrm{i}) = 1 + \alpha + \beta \mathrm{i} + \frac{\alpha^2}{2} + \alpha\beta\,\mathrm{i} - \frac{\beta^2}{2} + \frac{\alpha^3}{8} + \frac{3\alpha^2\beta\,\mathrm{i}}{8} - \frac{3\alpha\beta^2}{8} - \frac{\beta^3\,\mathrm{i}}{8}$$

$$+ \frac{\alpha^4}{24} + \frac{\alpha^3\beta\,\mathrm{i}}{6} - \frac{\alpha^2\beta^2}{4} - \frac{\alpha\beta^3\mathrm{i}}{6} + \frac{\beta^4}{24}$$

$$+ \frac{\alpha^5}{120} + \frac{\alpha^4\beta\mathrm{i}}{24} - \frac{\alpha^3\beta^2}{12} - \frac{\alpha^2\beta^3\mathrm{i}}{12} + \frac{\alpha\beta^4}{24} + \frac{\beta^5\mathrm{i}}{120}$$

$$+ \frac{\alpha^6}{864} + \frac{\alpha^5\beta\mathrm{i}}{144} - \frac{5\alpha^4\beta^2}{288} - \frac{5\alpha^3\beta^3\mathrm{i}}{216} + \frac{5\alpha^2\beta^4}{288} + \frac{\alpha\beta^5\mathrm{i}}{144} - \frac{\beta^6}{864}$$

$$+ \frac{\alpha^7}{8640} + \frac{7\alpha^6\beta\mathrm{i}}{8640} - \frac{21\alpha^5\beta^2}{8640} - \frac{35\alpha^4\beta^3\mathrm{i}}{8640} + \frac{35\alpha^3\beta^4}{8640} + \frac{21\alpha^2\beta^5\mathrm{i}}{8640} - \frac{7\alpha\beta^6}{8640} - \frac{\beta^7\mathrm{i}}{8640}$$

$$+ \frac{\alpha^8}{138240} + \frac{8\alpha^7\beta\mathrm{i}}{138240} - \frac{28\alpha^6\beta^2}{138240} - \frac{56\alpha^5\beta^3\mathrm{i}}{138240} + \frac{70\alpha^4\beta^4}{138240}$$

$$+ \frac{56\alpha^3\beta^5\mathrm{i}}{138240} - \frac{28\alpha^2\beta^6}{138240} - \frac{8\alpha\beta^7\mathrm{i}}{138240} + \frac{\beta^8}{138240} \ .$$

Dividing the real and the imaginary parts of the complex numbers in the right-hand-side of (71) lead to the following relation:

$$\textbf{(72)}\qquad \overline{\mathbf{R}}(\alpha+\beta i)=(1+\alpha+\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\frac{\alpha^3}{8}-\frac{3\alpha\beta^2}{8}+\frac{\alpha^4}{24}-\frac{\alpha^2\beta^2}{4}+\frac{\beta^4}{24}$$

$$+\frac{\alpha^5}{120}-\frac{\alpha^3\beta^2}{12}+\frac{\alpha\beta^4}{24}+\frac{\alpha^6}{864}-\frac{5\alpha^4\beta^2}{288}+\frac{5\alpha^2\beta^4}{288}-\frac{\beta^6}{864}$$

$$+\frac{\alpha^7}{8640}-\frac{21\alpha^5\beta^2}{8640}+\frac{35\alpha^3\beta^4}{8640}-\frac{7\alpha\beta^6}{8640}$$

$$+\frac{\alpha^8}{138240}-\frac{28\alpha^6\beta^2}{138240}+\frac{70\alpha^4\beta^4}{138240}-\frac{28\alpha^2\beta^6}{138240}+\frac{\beta^8}{138240})$$

$$+(\beta+\alpha\beta+\frac{\alpha^2\beta}{2}-\frac{\beta^3}{8}+\frac{3\alpha^3\beta}{8}-\frac{\alpha\beta^3}{6}+\frac{\alpha^4\beta}{24}-\frac{\alpha^2\beta^3}{12}+\frac{\beta^5}{120}$$

$$+\frac{\alpha^5\beta}{144}-\frac{5\alpha^3\beta^3}{216}+\frac{\alpha\beta^5}{144}+\frac{7\alpha^6\beta}{8640}-\frac{35\alpha^4\beta^3}{8640}+\frac{21\alpha^2\beta^5}{8640}-\frac{\beta^7}{8640}$$

$$+\frac{8\alpha^7\beta}{138240}-\frac{56\alpha^5\beta^3}{138240}+\frac{56\alpha^3\beta^5}{138240}-\frac{8\alpha\beta^7}{138240})i\ .$$

It is clear now that the values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are given by:

$$\textbf{(73)}\qquad \overline{\mathbf{A}}=1+\alpha+\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\frac{\alpha^3}{8}-\frac{3\alpha\beta^2}{8}+\frac{\alpha^4}{24}-\frac{\alpha^2\beta^2}{4}+\frac{\beta^4}{24}$$

$$+\frac{\alpha^5}{120}-\frac{\alpha^3\beta^2}{12}+\frac{\alpha\beta^4}{24}+\frac{\alpha^6}{864}-\frac{5\alpha^4\beta^2}{288}+\frac{5\alpha^2\beta^4}{288}-\frac{\beta^6}{864}$$

$$+\frac{\alpha^7}{8640}-\frac{21\alpha^5\beta^2}{8640}+\frac{35\alpha^3\beta^4}{8640}-\frac{7\alpha\beta^6}{8640}$$

$$+\frac{\alpha^8}{138240}-\frac{28\alpha^6\beta^2}{138240}+\frac{70\alpha^4\beta^4}{138240}-\frac{28\alpha^2\beta^6}{138240}+\frac{\beta^8}{138240}$$

and

$$(74)\qquad \overline{\mathbf{B}} = \beta + \alpha\beta + \frac{\alpha^2\beta}{2} - \frac{\beta^3}{8} + \frac{3\alpha^3\beta}{8} - \frac{\alpha\beta^3}{6} + \frac{\alpha^4\beta}{24} - \frac{\alpha^2\beta^3}{12} + \frac{\beta^5}{120}$$

$$+ \frac{\alpha^5\beta}{144} - \frac{5\alpha^3\beta^3}{216} + \frac{\alpha\beta^5}{144} + \frac{7\alpha^6\beta}{8640} - \frac{35\alpha^4\beta^3}{8640} + \frac{21\alpha^2\beta^5}{8640} - \frac{\beta^7}{8640}$$

$$+ \frac{8\alpha^7\beta}{138240} - \frac{56\alpha^5\beta^3}{138240} + \frac{56\alpha^3\beta^5}{138240} - \frac{8\alpha\beta^7}{138240} \ .$$

The requirement $\mathbf{R}(\tilde{\lambda}) \leq \mathbf{1}$ will be satisfied when (19) holds with the values of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ from (73) and (74).

The stability regions obtained by using the formulae derived in this sub-section are given in Fig.4.

4.5. Some explanatory remarks about the preparation of the plots

It is necessary to mention here that only **real arithmetic** is used in the programs by which the absolute stability regions given in Fig. 1 – Fig. 4 were drawn. More precisely, the formulae listed in the ends of the previous subsections are actually used in the programs. First and foremost it must be emphasized that formula (19) is always used to check whether the condition $\mathbf{R}(\tilde{\lambda}) \leq \mathbf{1}$ is satisfied of not. The quantities $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ that are needed in (19) are calculated by using the following formulae:

(a) Formulae (17) and (18) are used for the first-order one-stage Runge-Kutta method.

(b) Formulae (22) and (23) are used when the first-order one-stage Runge-Kutta method is combined with the Richardson Extrapolation.

(c) Formulae (22) and (23) are also used for the second-order two-stage Runge-Kutta method.

(d) Formulae (32) and (33) are used when the second-order two-stage Runge-Kutta method is combined with the Richardson Extrapolation.

(e) Formulae (35) and (36) are used for the third-order three-stage Runge-Kutta method.

(f) Formulae (52) and (53) are used when the third-order three-stage Runge-Kutta method is combined with the Richardson Extrapolation.

(g) Formulae (61) and (62) are used for the fourth-order four-stage Runge-Kutta method.

(h) Formula (73) and (74) are used when the fourth-order four-stage Runge-Kutta method is combined with the Richardson Extrapolation.

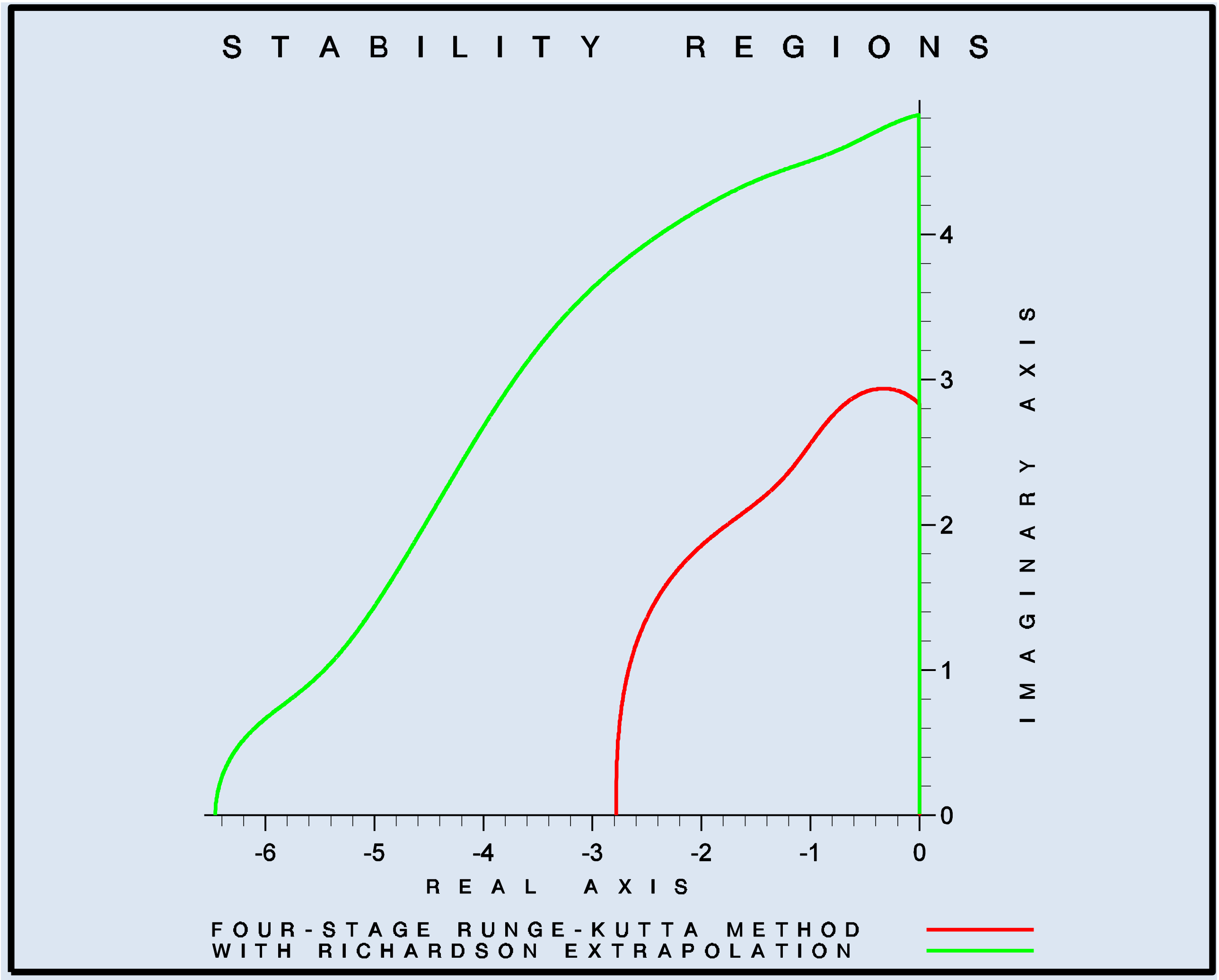

Figure 4

Stability regions of the original forth-order four-stage explicit method and the combination of the Richardson Extrapolation with this method.

5. Preparation of appropriate numerical examples

Two numerical examples will be defined and used in the following sections. These examples are represented with linear systems of ODEs with constant coefficients and are given in order to demonstrate the fact that the theoretical results related to the stability are valid also when the Richardson Extrapolation is applied. Each example contains three equations and its coefficient matrix has both real and complex eigenvalues. In the first example the real eigenvalue is dominant, while the complex eigenvalues put constraints on the stability of the computational process in the second example.

5.1. Numerical example with a large real eigenvalue

Consider the linear system of ordinary differential equations (ODEs) with constant coefficients given by

$$\textbf{(75)} \qquad y' = Ay, \qquad A \in R^{3\times 3}, \qquad y = (y_1, y_2, y_3)^T \in R^3 \qquad t \in [0, 13.1072].$$

Assume that the three components of the exact solution of (75) are given by

$$\textbf{(76)} \qquad y_1(t) = e^{-0.3t} \sin 8t + e^{-750t},$$

$$\textbf{(77)} \qquad y_2(t) = e^{-0.3t} \cos 8t - e^{-750t},$$

$$\textbf{(78)} \qquad y_3(t) = e^{-0.3t} (\sin 8t + \cos 8t) + e^{-750t}.$$

By differentiating the following equalities can be obtained from (76) – (77):

$$\textbf{(79)} \qquad \frac{dy_1(t)}{dt} = -0.3 e^{-0.3t} \sin 8t + 8 e^{-0.3t} \cos 8t - 750 e^{-750t},$$

$$\textbf{(80)} \qquad \frac{dy_2(t)}{dt} = -0.3 e^{-0.3t} \cos 8t - 8 e^{-0.3t} \sin 8t + 750 e^{-750t},$$

$$\textbf{(81)} \qquad \frac{dy_3(t)}{dt} = -0.3 e^{-0.3t} (\sin 8t + \cos 8t) + 8 e^{-0.3t} (\cos 8t - \sin 8t) - 750 e^{-750t}.$$

It is clear that the following three equations are to be satisfied if (76) – (78) are giving the exact solution of (75):

$$\textbf{(82)} \qquad \frac{dy_1(t)}{dt} = a_{11} y_1(t) + a_{12} y_2(t) + a_{13} y_3(t),$$

$$(83) \qquad \frac{dy_2(t)}{dt} = a_{21}\, y_1(t) + a_{22}\, y_2(t) + a_{23}\, y_3(t),$$

$$(84) \qquad \frac{dy_3(t)}{dt} = a_{31}\, y_1(t) + a_{32}\, y_2(t) + a_{33}\, y_3(t).$$

The following relationships can be obtained by inserting in (82) – (84) the values of the exact solution that are given by (76) – (78) and the values of its derivatives that are given by (79) – (81):

$$(85) \qquad -0.3\,e^{-0.3t}\sin 8t + 8\,e^{-0.3t}\cos 8t - 750\,e^{-750t}$$
$$= a_{11}\left(e^{-0.3t}\sin 8t + e^{-750t}\right) + a_{12}\left(e^{-0.3t}\cos 8t - e^{-750t}\right)$$
$$+ a_{13}\left(e^{-0.3t}(\sin 8t + \cos 8t) + e^{-750t}\right)$$

$$(86) \qquad -0.3\,e^{-0.3t}\cos 8t - 8\,e^{-0.3t}\sin 8t + 750\,e^{-750t}$$
$$= a_{21}\left(e^{-0.3t}\sin 8t + e^{-750t}\right) + a_{22}\left(e^{-0.3t}\cos 8t - e^{-750t}\right)$$
$$+ a_{23}\left(e^{-0.3t}(\sin 8t + \cos 8t) + e^{-750t}\right)$$

$$(87) \qquad -0.3\,e^{-0.3t}(\sin 8t + \cos 8t) + 8\,e^{-0.3t}(\cos 8t - \sin 8t) - 750\,e^{-750t}$$
$$= a_{31}\left(e^{-0.3t}\sin 8t + e^{-750t}\right) + a_{32}\left(e^{-0.3t}\cos 8t - e^{-750t}\right)$$
$$+ a_{33}\left(e^{-0.3t}(\sin 8t + \cos 8t) + e^{-750t}\right)$$

Relationships (85) – (87) can be rewritten as

$$(88) \qquad (-0.3 - a_{11} - a_{13})e^{-0.3t}\sin 8t + (8 - a_{12} - a_{13})e^{-0.3t}\cos 8t$$
$$+ (-750 - a_{11} + a_{12} - a_{13})e^{-750t} = 0,$$

$$(89) \qquad (-0.3 - a_{22} - a_{23})e^{-0.3t}\cos 8t + (-8 - a_{22} - a_{23})e^{-0.3t}\sin 8t$$
$$+ (750 - a_{21} + a_{22} - a_{23})e^{-750t} = 0,$$

$$(90) \qquad (-8.3 - a_{31} - a_{33})e^{-0.3t}\sin 8t + (7.7 - a_{32} - a_{33})e^{-0.3t}\cos 8t$$
$$+ (-750 - a_{31} + a_{32} - a_{33})e^{-750t} = 0.$$

The above equalities must be satisfied for all values of the time variable; therefore the coefficients before the functions must be equal to zero:

$$(91) \qquad -0.3 = a_{11} + a_{13}, \qquad 8 = a_{12} + a_{13}, \qquad -750 = a_{11} - a_{12} + a_{13}.$$

(92) $\mathbf{-0.3 = a_{22} + a_{23}}$, $\mathbf{-8 = a_{21} + a_{23}}$, $\mathbf{750 = a_{21} - a_{22} + a_{23}}$,

(93) $\mathbf{-8.3 = a_{31} + a_{33}}$, $\mathbf{7.7 = a_{32} + a_{33}}$, $\mathbf{-750 = a_{31} - a_{32} + a_{33}}$,

The elements of matrix can now be calculated from (91) – (93):

(94) $\mathbf{a_{11} = 741.4}$, $\mathbf{a_{12} = 749.7}$, $\mathbf{a_{13} = -741.7}$,

(95) $\mathbf{a_{21} = -765.7}$, $\mathbf{a_{22} = -758}$, $\mathbf{a_{23} = 757.7}$,

(96) $\mathbf{a_{31} = 725.7}$, $\mathbf{a_{32} = 741.7}$, $\mathbf{a_{33} = -734}$.

It should be mentioned here that the eigenvalues of matrix $\mathbf{A}$ from (75) are given by

(97) $\mathbf{\mu_1 = -750}$, $\mathbf{\mu_2 = -0.3 + 8i}$, $\mathbf{\mu_3 = -0.3 - 8i}$.

The absolute value of the real eigenvalue $\mathbf{\mu_1}$ is much larger than the absolute values of the two complex eigenvalues. This means that the computations will be stable when $|\mathbf{h\mu_1}|$ is smaller than the length of the stability interval on the real axis (this length is smaller than $\mathbf{3}$ for all four explicit Runge-Kutta methods studied in this paper).

The solution of the example presented in this section is given in Fig. 5.

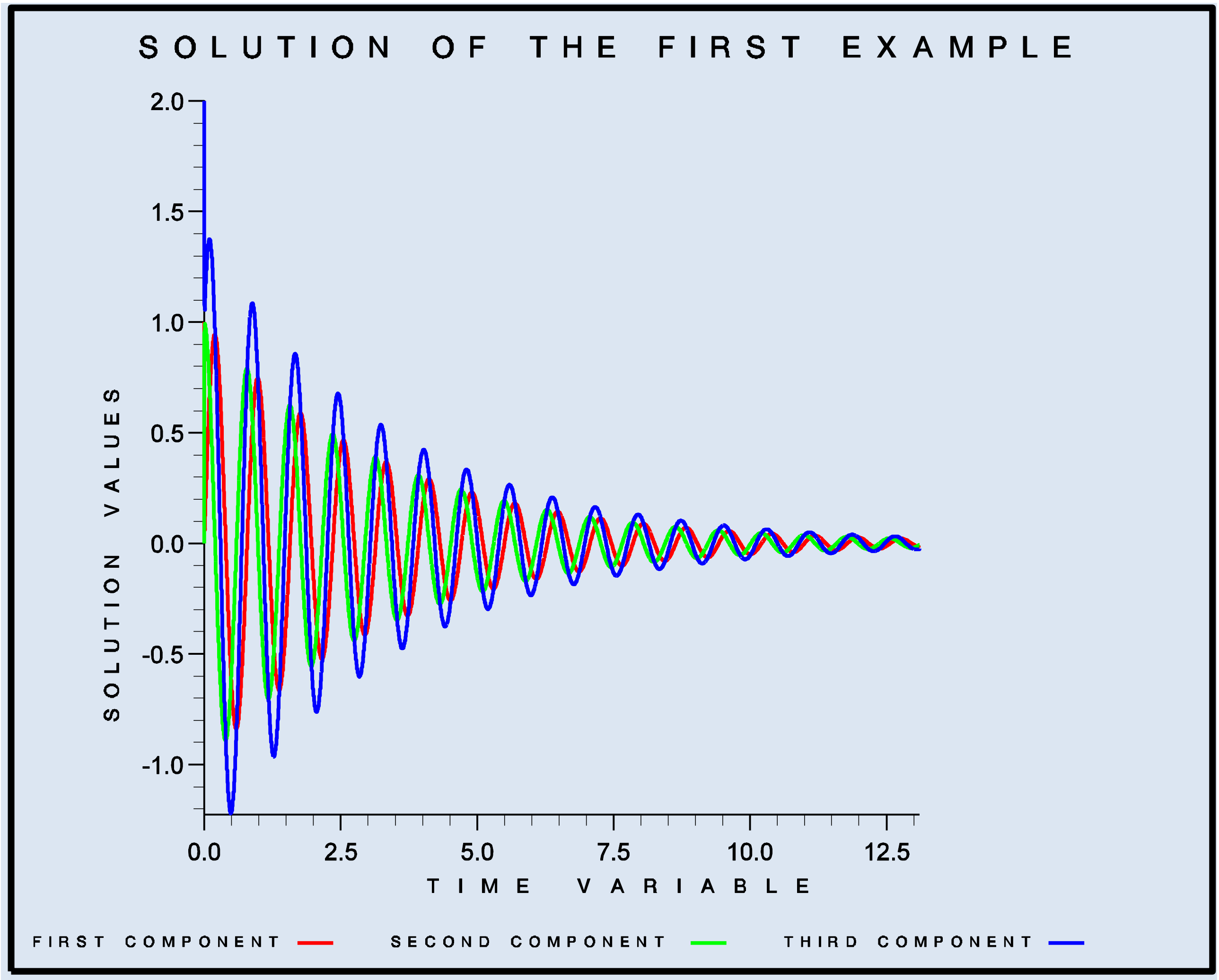

Figure 5

Plots of the three components of the solution of the system of ODEs defined by (75) where $t \in [0, 13.1072]$ and the elements of matrix $\mathbf{A}$ are given by (76) – (78). The real eigenvalue of matrix is much larger, in absolute value, than the two complex eigenvalues; see (97). The explicit first-order one-stage Ringe-Kutta method is used with $\mathbf{h} = 10^{-5}$ and the maximal error found during this run was approximately equal to $6.63 * 10^{-4}$.

5.2. Numerical example with a large complex eigenvalues

Consider the linear system of ordinary differential equations (ODEs) given by

$$\text{(98)} \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}, \qquad \mathbf{A} \in \mathbf{R}^{3\times3}, \qquad \mathbf{y} = (y_1, y_2, y_3)^T \in \mathbf{R}^3 \qquad t \in [0,\ 13.1072],$$

$$\mathbf{b} = \left(-4e^{-0.3t}\sin 4t,\ -8e^{-0.3}\sin 4t,\ 4e^{-0.3t}\sin 4t\right)^T, \quad \mathbf{y}(0) = (1, 3, 0)^T .$$

Assume that the three components of the exact solution of (98) are given by

$$\text{(99)} \qquad y_1(t) = e^{-750t} \sin 750t + e^{-0.3t} \cos 4t ,$$

$$\text{(100)} \qquad y_2(t) = e^{-750t} \cos 750t + 2e^{-0.3t} \cos 4t ,$$

$$\text{(101)} \qquad y_3(t) = e^{-750t} (\sin 750t + \cos 750t) - e^{-0.3t} \cos 4t .$$

The following equalities can be obtained by differentiating (99) – (101):

$$\text{(102)} \qquad \frac{dy_1(t)}{dt} = -750e^{-750t} \sin 750t + 750e^{-750t} \cos 750t - 0.3e^{-0.3t} \cos 4t - 4e^{-0.3t} \sin 4t ,$$

$$\text{(102)} \qquad \frac{dy_2(t)}{dt} = -750e^{-750t} \cos 750t - 750e^{-750t} \sin 750t - 0.6e^{-0.3t} \cos 4t - 8e^{0.3t} \sin 4t ,$$

$$\text{(103)} \qquad \frac{dy_3(t)}{dt} = -1500e^{-750t} \sin 750t + 0.3e^{-0.3t} \cos 4t + 4e^{-0.3t} \sin 4t .$$

It is clear that the following three equations are to be satisfied if (99) – (101) are giving the exact solution of (98):

$$\text{(104)} \qquad \frac{dy_1(t)}{dt} = a_{11} y_1(t) + a_{12} y_2(t) + a_{13} y_3(t) - 4e^{-0.3t} \sin 4t ,$$

$$\text{(105)} \qquad \frac{dy_2(t)}{dt} = a_{21} y_1(t) + a_{22} y_2(t) + a_{23} y_3 - 8e^{-0.3} \sin 4t(t) ,$$

$$\text{(106)} \qquad \frac{dy_3(t)}{dt} = a_{31} y_1(t) + a_{32} y_2(t) + a_{33} y_3(t) + 4e^{-0.3t} \sin 4t.$$

The following relationships can be obtained by inserting the values of the exact solution and its derivatives in (104) – (106):

$$\text{(107)} \qquad \begin{aligned} & -750e^{-750t} \sin 750t + 750e^{-750t} \cos 750t - 0.3e^{-0.3t} \cos 4t - 4e^{-0.3t} \sin 4t \\ & = a_{11}\left(e^{-750t} \sin 750t + e^{-0.3t} \cos 4t\right) + a_{12}\left(e^{-750t} \cos 750t + 2e^{-0.3t} \cos 4t\right) \\ & + a_{13}\left(e^{-750t} (\sin 750t + \cos 750t) - e^{-0.3t} \cos 4t\right) - 4e^{-0.3t} \sin 4t \end{aligned}$$

$$(108)\qquad -750e^{-750t}\cos 750\,t - 750\,e^{-750t}\sin 750\,t - 0.6e^{-0.3t}\cos 4t - 8e^{-0.3t}\sin 4t$$
$$= a_{21}\left(e^{-750t}\sin 750\,t + e^{-0.3t}\cos 4t\right) + a_{22}\left(e^{-750t}\cos 750\,t + 2e^{-0.3t}\cos 4t\right)$$
$$+ a_{23}\left(e^{-750t}(\sin 750\,t + \cos 750\,t) - e^{-0.3t}\cos 4\,t\right) - 8e^{-0.3t}\sin 4t$$

$$(109)\qquad -1500e^{-750t}\ \sin 750\,t + 0.3\,e^{-0.3t}\cos 4t + 4e^{0.3t}\sin 4t$$
$$= a_{31}\left(e^{-750t}\sin 750\,t + e^{-0.3t}\cos 4t\right) + a_{32}\left(e^{-750t}\cos 750\,t + 2e^{-0.3t}\cos 4t\right)$$
$$+ a_{33}\left(\ e^{-750t}(\sin 750\,t + \cos 750\,t) - e^{-0.3t}\cos 4t\right) + 4e^{0.3t}\sin 4\,t$$

Relationships (107) – (109) can be rewritten as

$$(110)\qquad (-750 - a_{11} - a_{13})e^{-750t}\sin 750t + (750 - a_{12} - a_{13})e^{-750t}\cos 750t$$
$$+ (-0.3 - a_{11} - 2a_{12} + a_{13})e^{-0.3t}\cos 4t = 0,$$

$$(111)\qquad (-750 - a_{22} - a_{23})e^{-750t}\cos 750t + (-750 - a_{21} - a_{23})e^{-750t}\sin 750t$$
$$+ (-0.6 - a_{21} - 2a_{22} + a_{23})e^{-0.3t}\cos 4t = 0,$$

$$(112)\qquad (-1500 - a_{31} - a_{33})e^{-750t}\sin 750t + (-a_{32} - a_{33})e^{-750t}\cos 750t$$
$$+ (0.3 - a_{31} - 2a_{32} + a_{33})e^{-0.3t}\cos 4t = 0.$$

The above equalities must be satisfied for all value of the time variable; therefore the coefficients before the functions must be equal to zero:

$$(113)\qquad -750 = a_{11} + a_{13}, \qquad 750 = a_{12} + a_{13}, \qquad -0.3 = a_{11} + 2a_{12} - a_{13}.$$

$$(114)\qquad -750 = a_{22} + a_{23}, \qquad -750 = a_{21} + a_{23}, \qquad -0.6 = a_{21} + 2a_{22} - a_{23},$$

$$(115)\qquad -1500 = a_{31} + a_{33}, \qquad 0 = a_{32} + a_{33}, \qquad 0.3 = a_{31} + 2a_{32} - a_{33}.$$

The elements of matrix $\mathbf{A}$ can now be calculated from (113) – (115):

$$(116)\qquad a_{11} = -937.575, \qquad a_{12} = 562.425, \qquad a_{13} = 187.575\ ,$$

$$(117)\qquad a_{21} = -187.65, \qquad a_{22} = -187.65\ , \qquad a_{23} = -562.35\ ,$$

(118) $\mathbf{a_{31} = -1124.925}$, $\mathbf{a_{32} = 375.075}$, $\mathbf{a_{33} = -375.075}$.

It should be mentioned here that the eigenvalues of matrix $\mathbf{A}$ from (98) are given by

(119) $\mu_1 = -750 + 750\mathbf{i}$, $\mu_2 = -750 - 750\mathbf{i}$, $\mu_3 = -0.3$.

The absolute value of each of the real eigenvalues μ_1 and μ_2 is much larger than the absolute values of the real eigenvalues. This mean that the computations will be stable when $|\mathbf{h}\mu_1|$ is inside of the absolute stability region of the numerical method under consideration.

The solution of the example presented in this section is given in Fig. 6.

6. Organization of the computations

The integration interval $[0, 13.1072]$ was divided into 128 equal sub-intervals and the accuracy of the results obtained by any of the selected numerical methods was evaluated at the end of each sub-interval. Let $\hat{\mathbf{t}}_j$, $\mathbf{j = 1, 2, \ldots, 128}$, be the end of any sub-interval. Then the following formula is used to evaluate the accuracy achieved at this point:

(120)
$$\mathbf{ERROR}_j = \frac{\sqrt{\left(y_1(\hat{t}_j) - \hat{y}_{1j}\right)^2 + \left(y_2(\hat{t}_j) - \hat{y}_{2j}\right)^2 + \left(y_3(\hat{t}_j) - \hat{y}_{3j}\right)^2}}{\max\left[\sqrt{\left(y_1(\hat{t}_j)\right)^2 + \left(y_2(\hat{t}_j)\right)^2 + \left(y_3(\hat{t}_j)\right)^2}, 1.0\right]} .$$

The values $\hat{\mathbf{y}}_{ij} \approx \mathbf{y}_i(\mathbf{t}_j)$, $\mathbf{i = 1, 2, 3}$, in (120) are approximations of the exact solution that are calculated by the selected numerical method.

The total error is computed as

(121)
$$\mathbf{ERROR} = \max_{j=1, 2, \ldots, 128} \left(\mathbf{ERROR}_j\right).$$

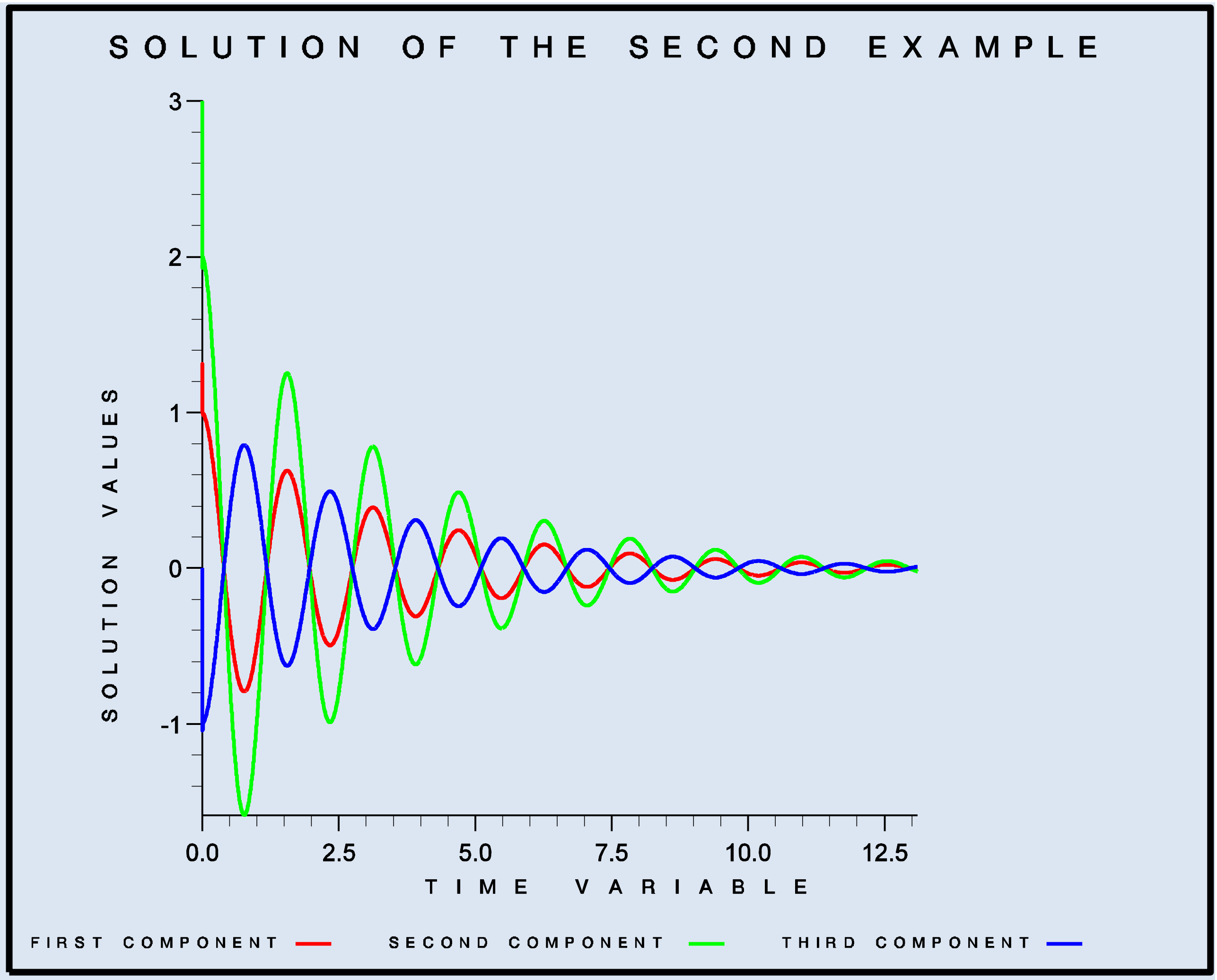

Figure 6

Plots of the three components of the solution of the system of ODEs defined by (98) where $\mathbf{t} \in [\ 0,\ 13.1072\ ]$ and the elements of matrix $\mathbf{A}$ are given by (116) – (118). The explicit Ringe-Kutta method is used with $\mathbf{h} = 10^{-5}$ and the maximal error found during this run was approximately equal to $4.03 * 10^{-5}$.

Ten runs were performed with every numerical method. The first run is carried out by using $\mathbf{h} = 0.00512$ and in each of the next nine runs the stepsize is halved (which leads automatically to performing twice more time-steps).

It should be mentioned here that all computations were carried out on the computers of the Centre for Scientific Computing at the Technical University of Denmark [9].

7. Particular numerical methods used in the experiments

There exists only one first-order one-stage explicit Runge-Kutta method (the Forward Euler Formula) which is given by

$$\textbf{(122)} \qquad y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) .$$

The situation changes when m-stage explicit Runge-Kutta methods of order p with $p = m$ and $p = 2, 3, 4$ are used. Then for each $p = m$ there exists a class of explicit Runge-Kutta methods. All methods from such a class have the same absolute stability region. Therefore, until now it was not necessary to specify which particular method was selected, because we were primarily interested in comparing the absolute stability regions of explicit Runge-Kutta methods with the corresponding absolute stability regions that are obtained when the Richardson Extrapolation is additionally used. However, it is necessary select one particular method from each class when numerical experiments are to be carried. The particular numerical methods that were used in the numerical solution of the example discussed in the previous sections are listed below.

The following method was chosen from the class of the second-order two-stage explicit Runge-Kutta methods:

$$\textbf{(123)} \qquad k_1 = f(t_{n-1}, y_{n-1}) ,$$

$$\textbf{(124)} \qquad k_2 = f(t_{n-1} + h, y_{n-1} + hk_1) ,$$

$$\textbf{(125)} \qquad y_n = y_{n-1} + \frac{1}{2}h(k_1 + k_2) .$$

The method selected from the class of the third-order three-stage explicit Runge-Kutta methods is defined as follows:

$$\textbf{(126)} \qquad k_1 = f(t_{n-1}, y_{n-1}) ,$$

(127) $$k_2 = f(t_{n-1} + \frac{1}{3}h, y_{n-1} + \frac{1}{3}hk_1),$$

(128) $$k_3 = f(t_{n-1} + \frac{2}{3}h, y_{n-1} + \frac{2}{3}hk_2),$$

(129) $$y_n = y_{n-1} + \frac{1}{4}h\left(k_1 + 3k_3\right).$$

One of the most popular methods from the class of the fourth-order four-stage explicit Runge-Kutta methods is chosen:

(130) $$k_1 = f(t_{n-1}, y_{n-1}),$$

(131) $$k_2 = f(t_{n-1} + \frac{1}{2}h, y_{n-1} + \frac{1}{2}hk_1),$$

(132) $$k_3 = f(t_{n-1} + \frac{2}{2}h, y_{n-1} + \frac{1}{2}hk_2),$$

(133) $$k_4 = f(t_{n-1} + h, y_{n-1} + hk_3),$$

(134) $$y_n = y_{n-1} + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right).$$

The numerical results which will be presented in the next section were obtained by using the above three particular explicit Runge-Kutta methods as well as the Forward Euler Formula.

8. Numerical results

Accuracy results, which are obtained when eight numerical methods for the solution of systems of ODEs are used, are given in Table 1 for the first example and in Table 3 for the second one. Convergence rates observed for the eight tested numerical methods are shown in Table 2 and Table 4 respectively.

Several major conclusions can immediately be drawn by investigating the results presented in Table 1 and Table 2:

(a) For the largest stepsize the explicit Runge-Kutta methods are producing unstable results when the first example is handled because $\left| \mathbf{h}\boldsymbol{\mu}_1 \right| = \mathbf{0.00512 * 750 = 3.84}$ is greater than the lengths of the stability intervals on the real axis for all four explicit Runge-Kutta methods (see Table 1). The situation is more complicated when the dominant eigenvalues are complex. Then it is more difficult to satisfy the restriction $\mathbf{R}(\tilde{\boldsymbol{\lambda}}) \leq \mathbf{1}$ and, therefore, also the stepsize $\mathbf{h = 0.00256}$ is giving problems (see Table 3). It should be noted that while the stability requirements are more stringent for the second problem, the accuracy achieved when the second problem is solved is considerably greater than the accuracy achieved in the solution of the first problem (under the assumption that the computations with both examples are stable). The reason for this behaviour can be explained by comparing the plots of the solution components given in Fig. 5 and Fig. 6. It is clearly seen that the solution components of the second solution are considerably smoother that those of the first example.

(b) The combination of the first-order one-stage Runge-Kutta method and the Richardson Extrapolation gives the same results as the second-order two-stage Runge-Kutta method. It is seen that the stability regions of these two numerical methods are also identical. The results indicate that this property holds not only for the Dahlquist test-example but also for linear systems of ODEs with constant coefficients. This property perhaps holds also for more general systems of ODEs.

(c) The results show that the calculated (as ratios of two consecutive error estimations) convergence rates of the Runge-Kutta method of order $\mathbf{p}$ are about $\mathbf{2^p}$ when the stepsize is reduced successively by a factor of two. For the combinations of the Runge-Kutta methods and the Richardson Extrapolation the corresponding convergence rates are $\mathbf{2^{p+1}}$ which means that the order of accuracy is increased by one.

(d) The great power of the Richardson Extrapolation is clearly demonstrated by the results given in Table 1. Consider the use of the first-order one-stage Runge-Kutta method together with the Richardson Extrapolation (denoted as **ERK1+R** in the table). The error estimation is $\mathbf{2.91 * 10^{-4}}$ when $\mathbf{h = 0.00128}$ and when $\mathbf{10240}$ time-steps are performed. Similar accuracy can be achieved by using $\mathbf{1310720}$ steps when the first-order one-stage Runge-Kutta method, **ERK1,** is used (i.e. the number of time-steps is increased by a factor grater than $\mathbf{100}$). Of course, for every step performed by the **ERK1** method, the **ERK1+R** method performs three steps (one large and two small). Even when this fact is taken into account (by multiplying the number of time-steps for **ERK1-R** by three), the **ERK1-R** is reducing the number of time-steps performed by **ERK1** by a factor greater than $\mathbf{30}$). The alternative is to use a method of higher order. However, such methods are more expensive and, what is perhaps much more important, a very cheap and reliable error estimation can be obtained when the Richardson Extrapolation is used. The situation is very similar also when the second example is treated.

	Stepsize	Steps	ERK1	ERK1+R	ERK2	ERK2+R	ERK3	ERK3+R	ERK4	ERK4+R
1	0.00512	2560	N.S.	N.S.	N.S.	2.39E-05	N.S.	6.43E-03	N.S.	4.49E-10
2	0.00256	5120	2.01E-01	4.22E-02	4.22E-02	2.99E-06	5.97E-06	7.03E-09	2.46E-08	1.41E-11
3	0.00128	10240	9.21E-02	2.91E-04	2.91E-04	3.73E-07	7.46E-07	4.40E-10	1.54E-09	4.39E-13
4	0.00064	20480	4.41E-02	7.27E-05	7.27E-05	4.67E-08	9.33E-08	2.75E-11	9.62E-11	1.37E-14
5	0.00032	40960	2.16E-02	1.82E-05	1.82E-05	5.83E-09	1.17E-08	1.72E-12	6.01E-12	4.29E-16
6	0.00016	81920	1.07E-02	4.54E-06	4.54E-06	7.29E-10	1.46E-09	1.07E-13	3.76E-13	1.34E-17
7	0.00008	163840	5.32E-03	1.14E-06	1.14E-06	9.11E-11	1.82E-10	6.71E-15	2.35E-14	4.19E-19
8	0.00004	327680	2.65E-03	2.84E-07	2.84E-07	1.14E-11	2.28E-11	4.20E-16	1.47E-15	1.31E-20
9	0.00002	655360	1.33E-03	7.10E-08	7.10E-08	1.42E-12	2.85E-12	2.62E-17	9.18E-17	4.09E-22
10	0.00001	1310720	6.66E-04	1.78E-08	1.78E-08	1.78E-13	3.56E-13	1.64E-18	5.74E-18	1.28E-23

Table 1

Accuracy results (error estimations) achieved when the first example from Section 6 is run by using eight numerical methods on a SUN computer by using quadruple precision. “N.S.” means that the numerical method is not stable for the stepsize used. “ERKi”, i=1,2,3,4, means explicit Runge-Kutta method of order p=i. “ERKi+R” refers to the explicit Runge-Kutta method of order p=i combined with the Richardson Extrapolation.

Run	Stepsize	Steps	ERK1	ERK1+R	ERK2	ERK2+R	ERK3	ERK3+R	ERK4	ERK4+R
1	0.00512	2560	N. A.	N. A.	N. A.	N. A.	N. A.	N. A.	N. A.	N. A.
2	0.00256	5120	N. A.	N. A.	N. A.	7.99	N. A.	very big	N. A.	31.84
3	0.00128	10240	2.18	145.02	145.02	8.02	8.00	15.98	15.97	32.12
4	0.00064	20480	2.09	4.00	4.00	7.99	8.00	16.00	16.01	32.04
5	0.00032	40960	2.04	3.99	3.99	8.01	7.97	15.99	16.01	31.93
6	0.00016	81920	2.02	4.01	4.01	8.00	8.01	16.07	15.98	32.01
7	0.00008	163840	2.01	3.98	3.98	8.00	8.02	15.95	16.00	31.98
8	0.00004	327680	2.01	4.01	4.01	7.99	7.98	15.97	15.99	31.98
9	0.00002	655360	1.99	4.00	4.00	8.03	8.00	16.03	16.01	32.03
10	0.00001	1310720	2.00	3.99	3.99	7.98	8.01	15.98	15.99	31.95

Table 2

Convergent rates (ratios of two consecutive error estimations from Table 1) observed when the first example from Section 6 is run by using eight numerical methods on a SUN computer by using quadruple precision. “N.A.” means that the convergence rate cannot be calculated (this happens either when the first run is performed or if the computations at the previous runs were not stable). “ERKi”, i=1,2,3,4, means explicit Runge-Kutta method of order p=i. “ERKi+R” refers to the explicit Runge-Kutta method of order p=i combined with the Richardson Extrapolation.

Run	Stepsize	Steps	ERK1	ERK1+R	ERK2	ERK2+R	ERK3	ERK3+R	ERK4	ERK4+R
1	0.00512	2560	N. S.	N. S.	N. S.	N. S.	N. S.	4.95E-02	N. S.	N. S.
2	0.00256	5120	N. S.	N. S.	N. S.	5.40E-08	N. S.	4.88E-13	N. S.	1.21E-17
3	0.00128	10240	2.37E-02	4.09E-06	6.81E-06	3.22E-11	1.54E-09	3.04E-14	7.34E-13	3.51E-19
4	0.00064	20480	2.58E-03	1.02E-06	1.70E-06	3.99E-12	1.92E-10	1.90E-15	4.59E-14	1.05E-20
5	0.00032	40960	1.29E-03	2.56E-07	4.26E-07	4.97E-13	2.40E-11	1.19E-16	2.87E-15	3.21E-22
6	0.00016	81920	6.45E-04	6.40E-08	1.06E-07	6.21E-14	3.00E-12	7.41E-18	1.79E-16	9.93E-24
7	0.00008	163840	3.23E-04	1.60E-08	2.66E-08	7.75E-15	3.75E-13	4.63E-19	1.12E-17	3.09E-25
8	0.00004	327680	1.61E-04	4.00E-09	6.65E-09	9.68E-16	4.69E-14	2.89E-20	7.00E-19	9.62E-27
9	0.00002	655360	8.06E-05	9.99E-10	1.66E-09	1.21E-16	5.86E-15	1.81E-21	4.38E-20	3.00E-28
10	0.00001	1310720	4.03E-05	2.50E-10	4.16E-10	1.51E-17	7.32E-16	1.13E-22	2.73E-21	9.36E-30

Table 3

Accuracy results (error estimations) achieved when the second example from Section 6 is run by using eight numerical methods on a SUN computer by using quadruple precision. "N.S." means that the numerical method is not stable for the stepsize used. "ERKi", i=1,2,3,4, means explicit Runge-Kutta method of order p=i. "ERKi+R" refers to the explicit Runge-Kutta method of order p=i combined with the Richardson Extrapolation.

Run	Stepsize	Steps	ERK1	ERK1+R	ERK2	ERK2+R	ERK3	ERK3+R	ERK4	ERK4+R
1	0.00512	2560	N. A.	N. A.	N. A.	N. A.	N. A.	N. A.	N. A.	N. A.
2	0.00256	5120	N. A.	N. A.	N. A.	N. A.	N. A.	1.01E+11	N. A.	N. A.
3	0.00128	10240	N. A.	N. A.	N. A.	167.70	N. A.	16.05	N. A.	34.47
4	0.00064	20480	9.96	4.01	4.01	8.07	8.02	16.00	15.99	33.43
5	0.00032	40960	2.00	3.98	3.99	8.03	8.00	15.97	15.99	32.71
6	0.00016	81920	2.00	4.00	4.02	8.00	8.00	16.06	16.03	32.33
7	0.00008	163840	2.00	4.00	3.98	8.01	8.00	16.00	15.98	32.14
8	0.00004	327680	2.01	4.00	4.00	8.01	8.00	16.02	16.00	32.12
9	0.00002	655360	2.00	4.00	4.01	8.07	8.00	15.97	15.98	32.07
10	0.00001	1310720	2.00	4.00	3.99	8.01	8.01	16.02	16.04	32.05

Table 4

Convergent rates (ratios of two consecutive error estimations from Table 3) observed when the second example from Section 6 is run by using eight numerical methods on a SUN computer by using quadruple precision. "N.A." means that the convergence rate cannot be calculated (this happens either when the first run is performed or if the computations at the previous runs were not stable). "ERKi", i=1,2,3,4, means explicit Runge-Kutta method of order p=i. "ERKi+R" refers to the explicit Runge-Kutta method of order p=i combined with the Richardson Extrapolation.

(e) It was necessary to apply quadruple precision only in order to be able to illustrate the ability of the methods to achieve very accurate results when their orders of accuracy are greater than three. However, it should be stressed here that in general it will not be necessary to apply quadruple precision, i.e. the use of the traditionally used double precision will nearly always be quite sufficient.

(f) The so-called **active implementation** (see [3] and [11]) of the Richardson Extrapolation is used in this paper. In this implementation, at each time-step the improved (by applying the Richardson Extrapolation) value $\mathbf{y_n}$ of the approximate solution is used in the calculation of $\mathbf{z_n}$ and $\mathbf{w_n}$. One can also apply another approach: the values of the previous approximations $\mathbf{z_{n-1}}$ and $\mathbf{w_{n-1}}$ can be used in the calculation of $\mathbf{z_n}$ and $\mathbf{w_n}$ respectively and after that to calculate the Richardson improvement $\mathbf{y_n = (2^p w_n - z_n)/(2^p - 1)}$. A **passive implementation** of the Richardson Extrapolation is obtained in this way (in this implementation the improved by the Richardson Extrapolation values of the approximations are calculated at every time-step, but not used in the further computation). It is clear that if the underlying method is absolutely stable for the two stepsizes $\mathbf{h}$ and $\mathbf{0.5h}$ then the passive implementation of the Richardson Extrapolation will also be absolutely stable. The results in the first lines of Table 1 and Table 3 show very clearly that the passive implementation of the Richardson Extrapolation **may fail** while the active one is successful. This may happen when the underlying method is not stable for the large stepsize, but the combined method is stable (due to the increased stability regions).

9. Major concluding remarks

Specific conclusions based on numerical results were drawn in the previous section. Some more general conclusions, based not only on numerical results, but also on the fact that the Richardson Extrapolation leads to a considerable improvement of the stability properties of the explicit Runge-Kutta methods when the number of stages is equal to the order of accuracy will be drawn below.

It is well known that the application of the Richardson Extrapolation leads to an improvement of the accuracy of the underlying numerical method, not only the explicit Runge-Kutta methods with $\mathbf{p = m}$, $\mathbf{m = 1, 2, 3, 4}$, see [3], [8] and [11]. In the present paper it was shown that the combined methods (any of the explicit Runge-Kutta methods with $\mathbf{p = m}$, $\mathbf{m = 1, 2, 3, 4}$, plus the Richardson Extrapolation) have additionally larger regions of absolute stability.

It must be mentioned here the improvement of the stability properties is not occurring in all implementations of the Richardson Extrapolation. Consider the well-known Trapezoidal Rule. It is well known that it is an A-stable numerical method. However, the combination of the Trapezoidal Rule with the active Richardson Extrapolation is not A-stable (see [2] and [3]). This example indicates that it is necessary to check carefully the stability properties of the methods arising when a given method for solving systems of ODEs is combined with the Richardson Extrapolation.

It will be interesting to investigate whether similar results (i.e. improved stability properties) can be achieved for some other methods for numerical solution of systems of ODEs. This will be a topic for future investigations.

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