$$I = \int_{E^{30}} \frac{4\ x_1\ x_3^2\ x_{21} \ldots x_{30}}{(1 + x_2 + x_4)^2}\ \mathrm{e}^{2x_1x_3+x_5+x_6+\ldots+x_{20}}$$

$$I \approx 3.24454045910515 \qquad (\textit{Mathematica})$$

Plain Monte Carlo estimate:

$$\overline{\theta_N} = \frac{1}{N} \sum_{i=1}^{N} \theta_i, \qquad \theta_i = f(\xi_i),$$

$\xi_1, \xi_2, \ldots, \xi_N$ are independent realizations of the random point ξ.
Empirical estimate of variance:

$$\mathrm{D}\theta = \frac{1}{N} \sum_{i=1}^{N} \theta_i^2 - \overline{\theta_N}^2.$$

Probable error:

$$r_N \approx 0.6745 \sqrt{\frac{\mathrm{D}\theta}{N}}.$$

Probability density function $p(\mathrm{x})$:

$$p(\mathrm{x}) = c|f(\mathrm{x})|$$

$$\theta_0 = \begin{cases} \dfrac{f(\mathrm{x})}{p(\mathrm{x})}, \ \mathrm{x} \in \Omega_+, \\ 0, \ \mathrm{x} \in \Omega_0. \end{cases}$$

$$\mathrm{D}\theta_0 = \int_{E^d} \frac{f^2(\mathrm{x})}{p^2(\mathrm{x})} p(\mathrm{x}) \mathrm{d}\mathrm{x} - I^2 = c\ I - I^2 = 0 \quad \left(c = \frac{1}{I} \right).$$

Probability distribution function $F(t)$

$$F(t) = \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_d} p(\mathrm{x}) \mathrm{dx}, \quad \mathrm{x} \in E^d$$

$$F(t) = \prod_{i=1}^{d} F_i(t_i), \text{ when } t_1, \dots, t_d \text{ are independent.}$$

How to obtain unknown $\xi_i, \ i = 1, \dots, d$?

$$F_i(\xi_i) = \gamma_i, \ i = 1, \dots, d \ \equiv \ \xi_i = F_i^{-1}(\gamma_i)$$

where

$$F_i(\xi_i) = \int_0^{\xi_i} p_i(x_i) \mathrm{d}x_i, \ p_i(x_i) = \int_{E^{d-1}} p(\mathrm{x}) \mathrm{d}x_1 \dots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \dots \mathrm{d}x_d.$$

$$I = \int_{E^7} \mathrm{e}^{1-(sin^2(\frac{\pi}{2}x_1)+sin^2(\frac{\pi}{2}x_2)+sin^2(\frac{\pi}{2}x_3))}\, arcsin\left(sin(1) + \frac{x_1 + \ldots + x_7}{200}\right)$$

Positive definite importance function:

$$p(x_1, \ldots, x_7) = \frac{1}{\eta}\, \mathrm{e}^{1-(sin^2(\frac{\pi}{2}x_1)+sin^2(\frac{\pi}{2}x_2)+sin^2(\frac{\pi}{2}x_3))},$$

where

$$\eta = \int_{E^7} p(\mathrm{x})\mathrm{dx} = \mathrm{e}\left[\int_0^1 \mathrm{e}^{-sin^2(\frac{\pi}{2}x)}\mathrm{dx}\right]^3.$$

Problem: How to simulate values of a 7-dimensional random variable $\Xi(\xi_1, \ldots, \xi_7)$ with a probability density function $p(\mathrm{x})$?

- Choose $p_{max} = \sup\limits_{\mathrm{x} \in E^7} p(\mathrm{x})$

- do until N points are accepted
  - generate $(\xi^{(i)}, \psi^{(i)}) \in E^{7+1}$
  - if $\left(\psi^{(i)} < \dfrac{p(\xi^{(i)})}{p_{max}}\right)$ then $\xi^{(i)}$ is accepted
    else reject the point.

Estimate of I:

$$\theta_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\xi^{(i)})}{p(\xi^{(i)})}, \quad \text{where} \xi^{(i)} \text{ is a point with a density } p(\mathrm{x}).$$