Tutorial on Monte Carlo

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1. **Random Variables**

(a) Find the mathematical expectation and the variance of the r.v. γ^3, where γ is a continuous uniformly distributed r.v. (c.u.d.r.v.) in $[0, 1]$.

(b) Consider an algorithm for generating a discrete r.v. ξ:

$$\xi = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix},$$

where $p_i = P\{\xi = x_i\} = \frac{1}{n}$ using a random number γ (γ is a c.u.d.r.v. in $[0, 1]$).

2. **Plain Monte Carlo Algorithms**

Consider a *Plain* Monte Carlo algorithm for computing

$$I = \int_0^1 \sqrt[5]{x}dx$$

using a r.v. θ with a constant density function $p(x) = const$. Show that the variance $D\theta$ is much smaller than I^2.

3. **Importance Sampling Monte Carlo Algorithms**

Consider a *Importance sampling Monte Carlo algorithm* for

$$I = \int_0^1 e^x dx.$$

Show that the variance of the algorithm $D\theta_0$ is zero.

4. **Symmetrization of the Integrand**

For calculating the integral

$$I = \int_0^1 e^x dx$$

apply Monte Carlo with a r.v. $\theta' = f_1(x)$ using *symmetrization of the integrand*: $f_1(x) = \frac{1}{2}[f(x) + f(a + b - x)]$. Show that $D\theta' << I^2$.

5. **Separation of the principle part**

Consider the integral

$$I = \int_0^1 f(x)p(x)dx,$$

where $f(x) = e^x$ and $p(x) = 1, \quad x \in [0, 1]$.
a) Find the value of I.
b) Consider a Monte Carlo algorithm using *separation of principle part* for $h(x) = x$ and show that $E\theta' = I$.
c) Show that $D\theta' << I^2$.

6. **Integration on Subdomain Monte Carlo Algorithm**

Consider the integral

$$I = \int\int_{\Omega} (2 - y) dxdy,$$

where $\Omega \equiv \{(x, y) : 0 \leq x \leq 1;\ \ 0 \leq y \leq 1 + \frac{x}{4}\}$. Consider an *integration on subdomain Monte Carlo algorithm* assuming that

$$I' = \int\int_{\Omega'} (2 - y) dxdy \quad \text{and} \quad c = \int\int_{\Omega'} p(x, y) dxdy,$$

where $\Omega' \equiv \{(x, y) : 0 \leq x \leq 1;\ \ 0 \leq y \leq 1\}$ using a r.v. θ' with a constant density function $p(x, y)$ (such that $\int\int_{\Omega} p(x, y) dxdy = 1$).

Show that:
a) $D\theta' << I^2$;
b) $D\theta' \leq (1 - c)D\theta$, where θ is the corresponding r.v. for the *plain* Monte Carlo integration.

1 Random Variables

(a)

$$E\gamma^3 = \int_0^1 x^3 p(x)dx \quad \text{Since} \int_0^1 p(x)dx = 1 \ \rightarrow p(x) \equiv 1.$$

$$E\gamma^3 = \int_0^1 x^3 dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}$$

$$D\gamma^3 = E\gamma^6 - (E\gamma^3)^2$$

$$E\gamma^6 = \int_0^1 x^6 dx = \left.\frac{x^7}{7}\right|_0^1 = \frac{1}{7}$$

$$D\gamma^3 = \frac{1}{7} - \left(\frac{1}{4}\right)^2 = \frac{1}{7} - \frac{1}{16} = \frac{9}{112} \approx 0.08036$$

(b)

$$\gamma \in [0, 1]$$

Divide the interval $[0, 1]$ into subintervals of length $\Delta_i = p_i = \frac{1}{n}$. Since $\sum_{i=1}^{n} p_i = 1$, $\sum_{i=1}^{n} \Delta_i = 1$

$$P\{\xi = x_i\} = P\{\gamma \in \Delta_i\} = \Delta_i = \frac{1}{n} = p_i.$$

Thus, $P\{\xi = x_i\} = \frac{1}{n}$.

2. **Plain Monte Carlo Algorithm**

$$I = \int_0^1 x^{1/5} dx; \quad p(x) \equiv 1 \text{ because } \int_0^1 p(x)dx = 1.$$

$$I = \int_0^1 x^{1/5} dx = \frac{5}{6} x^{\frac{6}{5}} \Big|_0^1 = \frac{5}{6};$$

$$\theta = \gamma^{1/5}, \quad \text{where } \gamma \text{ is a c.u.d.r.v. in } [0, 1]$$

$$D\theta = E\theta^2 - (E\theta)^2$$

$$E\theta^2 = \int_0^1 x^{2/5} dx = \frac{5}{7} x^{\frac{7}{5}} \Big|_0^1 = \frac{5}{7} \approx 0.7143$$

$$D\theta = \frac{5}{7} - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \frac{25}{36} = \frac{5}{252} \approx 0.01984 << I^2$$

3 Importance Sampling Algorithm

$$\hat{p} = c|f(x)| \quad \text{Since} \int_0^1 \hat{p}(x)dx = 1$$

$$c = \left[\int_0^1 |e^x| dx\right]^{-1} = \frac{1}{e-1}.$$

$$\hat{p} = \frac{1}{e-1} e^x$$

According to the definition of the *Importance Sampling Algorithm*

$$\theta_0(x) = \begin{cases} \frac{f(x)}{\hat{p}(x)}, & x \in \Omega_+, \\ 0, & x \in \Omega_0. \end{cases}$$

Thus,

$$\theta_0(x) = \frac{e^x}{e^x}(e-1) = e-1, \quad \text{for } x \in [0,1].$$

$$D\theta_0(x) = \int_0^1 (e-1)^2 dx - (e-1)^2 = 0$$

4 Symmetrization of the Integrand

Since

$$\int_0^1 p(x)dx = 1 \quad p(x) \equiv 1 \text{ and } f(x) = e^x.$$

$$I = \int_0^1 e^x dx = e^x|_0^1 = e - 1, \quad f_1(x) = \frac{1}{2}(e^x + e^{1-x})$$

The symmetrized random variable is

$$\begin{aligned}
\theta' &= \frac{1}{2}\left(e^\gamma + e^{1-\gamma}\right), \text{ where } \gamma \text{ is a c.u.d.r.v. in } [0,1]. \\
E\theta' &= \frac{1}{2}\int_0^1 \left(e^x + e^{1-x}\right) dx = \frac{1}{2}\left[e^x|_0^1 - e^{1-x}|_0^1\right] = \frac{1}{2}(e + e - 2) = e - 1 \\
D\theta' &= E(\theta')^2 - (E\theta')^2 \\
D\theta' &= \frac{1}{4}[2e - (e-1)(3e-5)] \approx 0.00392 << (e-1)^2 = I^2
\end{aligned}$$

5 Separation of the principle part

a)

$$I = \int_0^1 e^x dx = e^x\Big|_0^1 = e - 1,$$

$$I' = \int_0^1 h(x)p(x)dx = \int_0^1 x dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\theta' = f(x) - h(x) + I'; \quad \theta' = e^x - x + \frac{1}{2}$$

b)

$$E\theta' = \int_0^1 \left(e^x - x + \frac{1}{2}\right) dx = e^x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 + \frac{1}{2} = e - 1 - \frac{1}{2} + \frac{1}{2} = e - 1$$

c)

$$
\begin{aligned}
D\theta' &= E(\theta')^2 - (E\theta')^2 \\
E(\theta')^2 &= \int_0^1 \left(e^x - x + \frac{1}{2}\right)^2 dx \\
&= \int_0^1 \left[(e^x - x)^2 + e^x - x + \frac{1}{4}\right] dx \\
&= \int_0^1 \left(e^{2x} - 2e^x x + x^2\right) dx + \int_0^1 (e^x - x)dx + \frac{1}{4} \\
&= \frac{1}{2}\, e^{2x}\Big|_0^1 - 2J + \frac{x^3}{3}\Big|_0^1 + e^x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 + \frac{1}{4} \\
&= \frac{1}{2}(e^2 - 1) + \frac{1}{3} + (e - 1) - \frac{1}{2} + \frac{1}{4} - 2J \\
&= \frac{1}{2}(e^2 - 1) + (e - 1) - 2J + \frac{4 - 6 + 3}{12} \\
&= \frac{1}{2}(e^2 - 1) + (e - 1) - 2J + \frac{1}{12} \\
J &= \int_0^1 xe^x dx = \int_0^1 xd(e^x) = xe^x\Big|_0^1 - \int_0^1 e^x dx = e - (e - 1) = 1
\end{aligned}
$$

$$
\begin{aligned}
E(\theta')^2 = & \quad \frac{1}{2}(e^2 - 1) + (e - 1) - 2 + \frac{1}{12} \\
= & \quad \frac{1}{2}(e - 1)(e + 1 + 2) - \frac{23}{12} \\
= & \quad \frac{1}{2}(e - 1)(e + 3) - \frac{23}{12} \\
D\theta' = & \quad \frac{1}{2}(e - 1)(e + 3) - \frac{23}{12} - (e - 1)^2 \\
= & \quad \frac{1}{2}(e - 1)\left[(e + 3) - 2(e - 1)\right] - \frac{23}{12} \\
= & \quad \frac{1}{2}(e - 1)(5 - e) - \frac{23}{12}
\end{aligned}
$$

Thus,

$$
D\theta' = \quad (e - 1)(5 - e)/2 - \frac{23}{12} \approx 0.0437 << (e - 1)^2 = I^2.
$$

6 Integration on Subdomain Monte Carlo Algorithm

$$I = \int\int_{\Omega}(2-y)dxdy = \int_0^1 dx \int_0^{1+\frac{x}{4}}(2-y)dy = -\int_0^1 \left.\frac{(2-y)^2}{2}\right|_0^{1+\frac{x}{4}} dx$$
$$= \frac{1}{2}\int_0^1 \left[4 - \left(1-\frac{x}{4}\right)^2\right] dx = 2 - \frac{1}{2}\int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{16}\right) dx$$
$$= 2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{2\times 48} = \frac{3}{2} + \frac{1}{8} - \frac{1}{96} = \frac{13}{8} - \frac{1}{96} = \frac{155}{96} \approx 1.61458$$

To find $p(x,y)$ we have to take into account that

$$\int_0^1 \int_0^{1+1/4} p(x,y)dxdy = 1$$

Since $S_{\Omega} = 1\frac{1}{8} = \frac{9}{8}$, we have

$$p(x,y) = \frac{8}{9}, \quad (x,y) \in \Omega.$$

Then

$$I = \int\int_{\Omega} \underbrace{\frac{9}{8}(2-y)}_{f(x,y)} . \underbrace{\frac{8}{9}}_{p(x,y)} dxdy.$$

$$I' = \int_0^1 \int_0^1 (2-y)dxdy = \frac{3}{2};$$

$$c = \int_0^1 \int_0^1 p(x,y)dxdy = \int_0^1 \int_0^1 \frac{8}{9}dxdy = \frac{8}{9}.$$

Now one can define the r.v. θ' for *Integration on Subdomain* Monte Carlo:

$$\theta' = \quad I' + (1-c)f(\xi'),$$

where the random point $\xi' \in \Omega_1 = \Omega \setminus \Omega'$ has a density function

$$p_1(x,y) = \frac{p(x,y)}{1-c} = \frac{8/9}{1-8/9} = \frac{8.9}{9.1} = 8.$$

$$\begin{aligned}
D\theta' &= \quad E(\theta')^2 - (E\theta')^2 \quad \left(D\theta' = (1-c)^2 Df(\xi')\right) \\
Df(\xi') &= \quad E\left(f(\xi')\right)^2 - \left(Ef(\xi')\right)^2 \\
Ef(\xi') &= \quad \int_0^1 \int_1^{1+x/4} \frac{9}{8}(2-y).8dxdy = 9\int_0^1 dx \int_1^{1+x/4} (2-y)dy \\
&= \quad -\frac{9}{2}\int_0^1 (2-y)^2\Big|_1^{1+x/4} dx = \frac{9}{2}\int_0^1 \left[1-\left(1-\frac{x}{4}\right)^2\right] dx \\
&= \quad \frac{9}{2}\left[1-\int_0^1 \left(1-\frac{x}{2}+\frac{x^2}{16}\right) dx\right] \\
&= \quad \frac{9}{2}\left[1-1+\frac{1}{4}-\frac{1}{48}\right] = \frac{3}{2}.\frac{11}{16} = \frac{33}{32} = 1\frac{1}{33} \approx 1.03125
\end{aligned}$$

$$
\begin{aligned}
E(f(\xi'))^2 = & \int_0^1 \int_1^{1+x/4} \frac{81}{64}(2-y)^2.8dxdy = \frac{81}{8}\int_0^1 dx \int_1^{1+x/4} (2-y)^2 dy \\
= & -\frac{81}{24}\int_0^1 (2-y)^3\Big|_1^{1+x/4} dx = \frac{27}{8}\int_0^1 \left[1-\left(1-\frac{x}{4}\right)^3\right] dx \\
= & \frac{27}{8}\left[1-\int_0^1 \left(1-\frac{x}{4}\right)^3 dx\right] \\
= & \frac{27}{8}\left[1+\left(1-\frac{x}{4}\right)^4\Big|_0^1\right] = \frac{27}{8}\left[1+\left(\frac{3}{4}\right)^4-1\right] \\
= & \frac{27}{8}\times\frac{81}{256} = \frac{3^7}{2.4^5} \approx 1.0678
\end{aligned}
$$

$$Df(\xi') = \frac{2187}{2048} - \frac{1089}{1024} = \frac{9}{2048} \approx 0.0043945$$

$$D\theta' = (1-c)^2 Df(\xi') = \frac{1}{9^2} \times \frac{9}{2048} = \frac{1}{18432} \approx 5.4253.10^{-5} << l^2$$

For the *Plain Monte Carlo Algorithm* we have:

$$\begin{aligned}
\theta = & \quad f(\xi) = \frac{9}{8}(2-\xi) \\
E\theta = & \quad \int\int_{\Omega} \underbrace{\frac{9}{8}(2-y)}_{f(x,y)} . \underbrace{\frac{8}{9}}_{p(x,y)} dxdy = \frac{155}{96} \approx 1.61458
\end{aligned}$$

$$
\begin{aligned}
E\theta^2 &= \iint_\Omega \frac{81}{64}(2-y)^2 . \frac{8}{9} dxdy \\
&= \frac{9}{8}\int_0^1 dx \int_0^{1+x/4} (2-y)^2 dy = -\frac{9}{24}\int_0^1 \left.(2-y)^3\right|_0^{1+x/4} dx \\
&= \frac{3}{8}\int_0^1 \left[8-\left(1-\frac{x}{4}\right)^3\right] dx = 3-\frac{3}{8}\int_0^1 \left(1-\frac{x}{4}\right)^3 dx \\
&= 3+\frac{3}{8}\left.\left(1-\frac{x}{4}\right)^4\right|_0^1 = 3+\frac{3}{8}\left[-1+\left(\frac{3}{4}\right)^4\right] \\
&= 3-\frac{3}{8}+\frac{3^5}{8.4^4} = \frac{21}{8}+\frac{3^5}{8.4^4} = \frac{21.4^4+3^5}{8.4^4} \\
&= \frac{5376+243}{8.4^4} = \frac{5619}{2048} \approx 2.74365
\end{aligned}
$$

Thus

$$D\theta = \frac{5619}{8.4^4} - \frac{155^2}{(4^2.3.2)^2} = \frac{2521}{18432} \approx 0.136773.$$

Now, it's easy to check that

$$D\theta' \leq (1-c)D\theta.$$

Indeed,

$$5.4253.10^{-5} \leq \frac{1}{9} \times 0.136773.$$