Optimal Approximation Of The Solutions Of The Stochastic Differential Equations With Irregular Coefficients

Paweł Przybyłowicz

We consider pointwise approximation of the solution of a scalar stochastic differential equation (SDE) of the form

$$\begin{cases} dX(t) = \sigma_1(t)a(X(t))dt + \sigma_2(t)b(X(t))dW(t), & t \in [0,T], \\ X(0) = \eta, \end{cases} \tag{1}$$

where $a, b : \mathbf{R} \to \mathbf{R}$ are at least Lipschitz in $\mathbf{R}$ and we allow for coefficients $\sigma_1, \sigma_2 : [0,T] \to \mathbf{R}$ to be singular.

We first consider the regular case, when σ_1, σ_2 belong to the class of Hölder continuous functions with Hölder exponent $\varrho \in (0,1]$. In the additive noise case ($b \equiv const$) we show that the classical Euler algorithm X^E has the optimal worst case error $\Theta(n^{-\varrho})$. In the multiplicative noise case we prove that the Euler algorithm X^E has the error $O(n^{-\min\{1/2,\varrho\}})$, which is optimal if $\varrho \in (0,1/2]$.

In the singular case, we consider a class of functions σ_1, σ_2 that are piecewise Hölder continuous, except for a finite number of unknown singular points. We investigate error of the classical Euler algorithm X^E and conclude that only singularities of σ_2 have influence on its accuracy. This will allow us to show that the algorithm X^E has the error $O(n^{-\min\{1/2,\varrho\}})$, both in the additive and multiplicative noise case. Moreover, we show that any algorithm which does not locate singularity of the coefficient σ_2 has the error $\Omega(n^{-\min\{1/2,\varrho\}})$. This bound holds even if there is at most one unknown singular point. Hence, we turn to the class of algorithms which adaptively locate unknown singularities of σ_2. In the additive noise case, under certain conditions about singular points, we show the construction of Euler–type algorithms with adaptive grid which preserve the optimal error $\Theta(n^{-\varrho})$, known from the regular case.