The Randomized Setting for Linear Multivariate Problems Defined over L_2

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We study the approximation of a continuous linear operator $S_d : F_d \to G_d$, where F_d is a weighted L_2 space of d-variate functions and G_d is an arbitrary Hilbert space. The approximation is considered in the randomized setting for algorithms from the class $\Lambda^{\rm std}$ if they use only function values, or from the class $\Lambda^{\rm all}$ if they use arbitrary linear functionals. It is known that randomization for the class $\Lambda^{\rm all}$ does not help in the sense that it yields essentially the same results as in the worst case setting for the class $\Lambda^{\rm all}$. Our main goal is to verify what is the power of the class $\Lambda^{\rm std}$ in the randomized setting and when it matches the power of $\Lambda^{\rm all}$ in the randomized or, equivalently, in the worst case setting.

In particular, we prove that the nth minimal randomized error of algorithms that use at most n function values at randomized points for approximation of a continuous linear functional S_d is

$$e^{\rm ran}(n, S_d; \Lambda^{\rm std}) = \frac{a_n \|S_d\|}{\sqrt{n}} \quad \text{with} \quad a_n \in [1/2, 1].$$

Hence, all non-zero continuous functionals S_d admit essentially the same error bounds and the speed of convergence in the randomized setting.

On the other hand, if the singular values of S_d behave as $\Theta(n^{-p})$ then the nth minimal errors for the classes $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ behave respectively as

$$\begin{aligned} e^{\rm ran}(n, S_d; \Lambda^{\rm std}) &= \Theta\left(n^{-\min(1/2,p)}\right), \\ e^{\rm ran}(n, S_d; \Lambda^{\rm all}) &= \Theta\left(n^{-p}\right). \end{aligned}$$

Hence, for $p \le 1/2$ the power of $\Lambda^{\rm std}$ is the same as the power of $\Lambda^{\rm all}$.

We also study various notions of tractability, i.e., how the minimal randomized errors depend on the number d of variables. It turns out that tractability for classes $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ are equivalent but sometimes with different exponents of tractability.