Numerical Analysis of Multilevel Monte Carlo for Scalar Jump-diffusion SDEs

Yuan Xia

joint work with Mike Giles

yuan.xia@oxford-man.ox.ac.uk
Oxford University Mathematical Institute
Oxford-Man Institute of Quantitative Finance

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Outline

This talk is based on Yuan Xia's joint work with Mike Giles: , . Due to time limits, we recommend interested readers to refer to the draft paper [XG11b] on numerical analysis and numerical results in [XG11a], [Xia11]. The methodology we use is the same to [Gil07].

Multilevel Approach for geometric Brownian SDEs

Given a scalar SDE driven by a Brownian diffusion

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t),$$

to estimate $\mathbb{E}[P] := \mathbb{E}[f(S(T))]$ where the path-dependent payoff P can be approximated by $\widehat{P}_\ell$ using 2^ℓ uniform timesteps, we use

$$\mathbb{E}[\widehat{P}_\ell] = \mathbb{E}[\widehat{P}_0] + \sum_{l=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}].$$

$\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ is estimated using N_ℓ simulations with same $W(t)$ for both $\widehat{P}_\ell$ and $\widehat{P}_{\ell-1}$,

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell^{(i)} - \widehat{P}_{\ell-1}^{(i)} \right)$$

Numerical and analysis results for variance of ML estimator using Milstein scheme

Option	Plain diffusion		Jump-diffusion	
	numerics	analysis	numerics	analysis
Lipschitz	$O(h^2)$	$O(h^2)$	$O(h^2)$	$O(h^2)$
Asian	$O(h^2)$	$O(h^2)$	$O(h^2)$	$O(h^2)$
lookback	$O(h^2)$	$o(h^{2-\delta})$	$O(h^2)$	$o(h^{2-\delta})$
barrier	$O(h^{3/2})$	$o(h^{3/2-\delta})$	$O(h^{3/2})$	$o(h^{1-\delta})$
digital	$O(h^{3/2})$	$o(h^{3/2-\delta})$	$O(h^{3/2})$	$o(h^{3/2-\delta})$

Table: numerical V_ℓ convergence rate for const rate jump-diffusion SDEs and analysis results proved.

MLMC Theorem

Theorem

Let P be a functional of the solution of a stochastic o.d.e., and $\widehat{P}_\ell$ the discrete approximation using a timestep $h_\ell = 2^{-\ell}\, T$.

there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, with computational complexity (cost) C_ℓ, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that

i) $\left|\mathbb{E}[\widehat{P}_\ell - P]\right| \leq c_1\, h_\ell^\alpha$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & l = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & l > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \leq c_2\, N_\ell^{-1} h_\ell^\beta$

iv) $C_\ell \leq c_3\, N_\ell\, h_\ell^{-1}$

Theorem

there exists a positive constant c_4 such that for any $\varepsilon < e^{-1}$ there are values L and N_ℓ for which the multilevel estimator

$$\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_\ell,$$

has Mean Square Error $MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > 1, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4\, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Jump-diffusion SDEs

To capture the characteristics of fat-tail return distribution and the volatility smile effect, Merton introduced jump-diffusion SDEs to model stock price:

$$\mathrm{d}S(t) = a(S(t-), t)\,\mathrm{d}t + b(S(t-), t)\,\mathrm{d}W_t + c(S(t-), t)\,\mathrm{d}J_t, \quad (1)$$

where the jump item J_t is a compound Poisson process

$\sum_{i=1}^{N(t)}(Z_i - 1)$, the jump magnitude Z_i satisfies some probability

distribution, and $N(t)$ is a Poisson process with intensity $\lambda = \lambda(S_t, t)$, independent of the Brownian motion.

Assumptions on the SDEs

We assume that the drift function $a \in C^{1,1}(\mathbb{R}\times\mathbb{R}^+)$, volatility function $b \in C^{2,1}(\mathbb{R}\times\mathbb{R}^+)$ and jump coefficient c a measurable function on $\mathbb{R}\times\mathbb{R}^+ \times E$ satisfy the following conditions where $L_0 \equiv \partial/\partial t + a\,\partial/\partial S$ and $L_1 \equiv b\,\partial/\partial S$.

- A1 (uniform Lipschitz condition): for $\varphi = a, b, L_1 b, c, \lambda$, there exists K_1 such that
$$|\varphi(x,t) - \varphi(y,t)| \le K_1\,|x-y|$$
- A2 (linear growth bound): for $\varphi = a, b, L_0 a, L_1 a, L_0 b, L_1 b, L_0 L_1 b, L_1 L_1 b, c$, there exists K_2 such that
$$|\varphi(x,t)| \le K_2\,(1+|x|)$$
- A3 (additional Lipschitz condition): there exists K_3 such that
$$|b(x,t) - b(x,s)| \le K_3\,(1+|x|)\,\sqrt{|t-s|}$$

A jump-adapted Milstein discretisation

In the constant rate case, we use Platen's jump-adapted approximation [Pla82], which combines jump times $\mathbb{J} = \{\tau_1, \tau_2, \ldots, \tau_m\}$ and the usual fixed-time grid.
The jump-adapted Milstein scheme on level ℓ, for $n = 0, \ldots, n_T - 1 := N(T) + 2^\ell - 1$ is

$$\widehat{S}^-_{n+1} = \widehat{S}_n + a_n\, h_n + b_n\, \Delta W_n + \tfrac{1}{2}\, b'_n\, b_n\, (\Delta W_n^2 - h_n), \tag{2}$$

$$\widehat{S}_{n+1} = \begin{cases} \widehat{S}^-_{n+1} + c(\widehat{S}^-_{n+1}, t_{n+1})(Z_i - 1), & \text{when } t_{n+1} = \tau_i; \\ \widehat{S}^-_{n+1}, & \text{otherwise.} \end{cases}$$

Strong convergence of the scheme

[BLP05] defines a continuous time interpolant of (2):

$$\widehat{S}_{KP}(t) = \widehat{S}_n + a_n\,(t - t_n) + b_n\,(W_t - W_n) + \frac{1}{2}\, b_n'\, b_n \left((W_t - W_n)^2 - (t - t_n) \right), \quad (3)$$

for $t_n \leq t \leq t_{n+1}$ and proves

Theorem

Provided the assumptions A1-A3 are satisfied, then for $m = 2$ there exists a constant C_m such that for the solution to (1) and (3)

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |S(t) - \widehat{S}_{KP}(t)|^m \right] < C_m\, h^m, \ \mathbb{E}\left[\sup_{0\leq t\leq T} |\widehat{S}_{KP}(t)|^m \right] < C_m.$$

Their result can be generalized to any integer $m \geq 2$ with the same methodology.

Lipschitz payoff

For the case where the payoff is a Lipschitz function of the value of the underlying asset at maturity (e.g. European vanilla option),

$$P = f(S(T)).$$

In the numerical discretisation, the estimator is defined by

$$\widehat{P} = f(\widehat{S}_{n_T}),$$

By the strong convergence theorem, we have:

Theorem

This approximation for Lipschitz payoffs has $V_\ell = O(h_\ell^2)$.

Brownian Bridge interpolant

By approximating the diffusion part as having constant drift and volatility within each timestep, we define the interpolant

$$\widehat{S}(t) = \widehat{S}_n + v_n\,(\widehat{S}^-_{n+1} - \widehat{S}_n) + b_n\,\Big(W(t) - W_n - v_n\,(W_{n+1} - W_n)\Big) \tag{4}$$

where $v_n \equiv (t - t_n)/h_n$. We can then use Brownian Bridge results to construct multilevel estimators for various payoffs.

Theorem

If $\widehat{S}(t)$ is the interpolant defined by (4) and $\widehat{S}_{KP}(t)$ is the Kloeden-Platen interpolant defined by (3) then

$$\mathbb{E}\left[\sup_{[0,T]}\left|\hat{S}(t) - \hat{S}_{KP}(t)\right|^m\right] = O((h\log h)^m),$$

Lookback option

The payoff is: $P = \exp(-rT)\,(S(T) - \min_{0\le t\le T} S(t))$.
The Brownian interpolated jump-adapted Milstein MC estimator would be $\widehat{P} = \exp(-rT)\left(\widehat{S}(T) - \min_{0\le t\le T}\widehat{S}(t)\right)$.
Using Hölder inequality and Theorem 3 we obtain

Theorem

The multilevel approximation for a lookback option which is a uniform Lipschitz function of $S(T)$ and $\inf_{[0,T]} S(t)$ has $V_\ell = o(h_\ell^{2-\delta})$ for any $\delta > 0$.

Barrier option

Payoff is $P = f(S(T))\, \mathbb{1}_{\{M_T > B\}}$, $M_T = \min_{0\le t\le T} S(t)$.
One estimator is $\widehat{P} = f(\widehat{S}(T))\mathbb{1}_{\{\hat{M}_T > B\}}$, $\hat{M} = \min_{0\le t\le T} \widehat{S}(t)$.

Theorem

Provided $\inf_{[0,T]} S(t)$ *has a bounded density in the neighbourhood of* B*, then the multilevel estimator for a down-and-out barrier option has variance* $V_\ell = o(h_\ell^{1-\delta})$ *for any* $\delta > 0$*.*

A better estimator uses expectation conditional on path values at two endpoints of each interval. Numerical evidence suggest $V_\ell = o(h_\ell^{3/2-\delta})$, but we cannot get an analytical bound at the moment.

Digital option

The digital payoff is $P = \mathbb{1}_{\{S(T)>K\}}$, we use conditional expectation to reduce variance of $\widehat{P} = \mathbb{1}_{\{\widehat{S}(T)>K\}}$:

$$\widehat{P}^f_\ell = \mathbb{E}\left[\mathbb{1}_{\left\{\widehat{S}^f_{n_T}>K\right\}} \mid \widehat{S}^f_{n_T-1}\right].$$

Distinguishing three circumstances of last jump times we can prove:

Theorem

Provided $b(K,T) \neq 0$, and $S(t)$ has a bounded density in the neighborhood of K, then the multilevel estimator for a digital option has variance $V_\ell = o(h_\ell^{3/2-\delta})$ for any $\delta > 0$.

State-dependent intensity

The case of bounded state-dependent intensity is discussed by Glasserman & Merener [GM04] who use a thinning approach.

The idea is to use a constant rate λ_{sup} Poisson process to generate candidate jump times. This is a superposition of two jump process:

1. Desired process with rate $\lambda(S(\tau-), \tau)$;
2. Zero-jump process with rate $\lambda_{\text{sup}} - \lambda(S(\tau-), \tau)$.

Thinning Algorithm

Hence we can have acceptance-rejection procedure:

1. Generate the jump-adapted time grid using Poisson process with constant rate λ_{sup};
2. Simulate each interval of time grid using appropriate discretisation scheme;
3. When the endpoint τ is a jump time, generate a uniform random number $U \sim [0, 1]$
   1. If $p = \dfrac{\lambda(S(\tau-), \tau)}{\lambda_{\text{sup}}} > U$, accept τ add jump to the state value;
   2. Otherwise turn to step 2.

Adopting multilevel

When using simple thinning, jump candidate selection may differ on fine and coarse grids, enlarging variance.
To circumvent this, we change the measure so that the probability of acceptance is the same on fine and coarse path:

$$\mathbb{E}_f[\widehat{P}_\ell] - \mathbb{E}_c[\widehat{P}_{\ell-1}] = \mathbb{E}_Q[\widehat{P}_\ell \prod_\tau R^f_\tau - \widehat{P}_{\ell-1} \prod_\tau R^c_\tau].$$

In both fine and coarse path, the acceptance probability for a candidate jump under the measure Q is defined to be $\frac{1}{2}$. The corresponding Radon-Nikodym derivatives are

$$R_\tau = \begin{cases} 2p_\tau, & \text{if } U < \frac{1}{2}; \\ 2(1-p_\tau), & \text{if } U \geq \frac{1}{2}, \end{cases} \quad p_\tau = \frac{\lambda(S(\tau-),\tau)}{\lambda_{\sup}}.$$

Variance order, change of measure

We can rewrite the estimator to bound two parts:

$$\left(\widehat{P}_\ell - \widehat{P}_{\ell-1}\right)\prod_{i=1}^{N_T} R_i^c + \widehat{P}_\ell \sum_{j=1}^{N_T}\left(R_j^f - R_j^c\right)\prod_{i=1}^{j-1} R_i^f \prod_{i=j+1}^{N_T} R_i^c.$$

The bound for the first one is implied from bound in constant rate case and the Radon-Nikodym derivative part is bounded by strong convergence result.

Theorem

Let $\widehat{P}_\ell, \widehat{P}_{\ell-1}$ be ML estimator for constant rate, then for arbitrary $p > 1$, the multilevel estimator for a thinning with change of measure has the variance

$$V_\ell \leq C_p \left(\mathbb{E}\left[\left(\widehat{P}_l - \widehat{P}_{l-1}\right)^{2p}\right]\right)^{1/p} + O\left((h_\ell \log h_\ell)^2\right).$$

Weak convergence of the estimator

To fulfill the complexity theorem, we also need to justify weak convergence of the estimator for each payoffs.

- Constant rate case
  - Lipschitz case can be implied from strong convergence of the scheme.
  - Lookback case is derived from the bound of discrepancy between KP interpolant and Brownian interpolant, discrepancy between KP interpolant and analytic solution.
  - For Barrier/Digital payoff we assume boundedness of density of $S_{\min}$ at B/density of S_T at K and use extreme value theory argument.
- Path-dependent rate case, we split the estimator into two parts and bound them accordingly.

Conclusion

- Multilevel jump-adapted approach for scalar jump-diffusion SDEs is supported through numerical analysis to bound the convergence of the multilevel correction variance.
- We bound the variance for European call, Asian, lookback, barrier and digital options in the constant jump rate case.
- We extend the analysis to cover bounded state-dependent intensities.
- We also prove weak convergence for each payoff.
- Consequently, $O(\epsilon^{-2})$ computational complexity is proved for all cases except for the barrier option for which the best that can currently be proved is $o(\epsilon^{-2-\delta})$ for any strictly positive δ.

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