Distribution Properties of Generalized van der Corput Sequences

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Borovets, August 30, 2011

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Research supported by the Austrian Science Foundation, project S9609

Outline

- Basic definitions
  - Uniform distribution modulo 1
  - Discrepancy
  - Examples
- Generalized van der Corput sequences
  - Idea
  - More generalizations
  - Properties
- Work of H. Faure
  - Basic tools
  - Main Theorems
  - Our results

Discrepancy function

Let $\omega = (x_n)_{n\geq 1}$, $x_n \in [0,1)$, be a one-dimensional infinite sequence. For $N \geq 1$ and for an interval $J := [\alpha, \beta) \subseteq [0,1)$,

$$A(J, N, \omega)$$

gives the number of indices $n \leq N$, for which $x_n \in J$. We call

$$\Delta(J, N, \omega) := \frac{A(J, N, \omega)}{N} - \lambda(J).$$

the discrepancy function of the interval J, where $\lambda(J)$ denotes the length of the interval J respectively the Lebesgue measure.

Uniform distribution modulo 1

Definition

A sequence $\omega = (x_n)_{n\geq 1}$ is said to be **uniformly distributed modulo 1** (for short u.d. mod 1) if for every subinterval $J \subseteq [0,1)$ we have

$$\lim_{N\to\infty} \frac{A(J,N,\omega)}{N} = \lambda(J).$$

Discrepancy

Definition

Let $x_1, \ldots, x_N$ be a finite sequence of real numbers and let I be the unit interval. The number

$$D_N = D_N(x_1, \ldots, x_N) = \sup_{J \subseteq I} |\Delta(J, N, \{x_1, \ldots, x_N\})|$$

is called the **extreme discrepancy** of the given sequence. For an infinite sequence ω of real numbers $D_N(\omega)$ should denote the discrepancy of the initial segment formed by the first N terms of ω.

(star discrepancy D_N^*: replace J by $J^* = [0, \alpha)$ with $0 \leq \alpha \leq 1$)

Diaphony

Another measure of the irregularities of distribution of an infinite sequence is due to Zinterhof (1976)

Definition

The diaphony F of the first N points of ω is defined

$$F_N(\omega) := \left(2 \cdot \sum_{m=1}^{\infty} \frac{1}{m^2} \left| \frac{1}{N} \sum_{n=1}^{N} \exp^{2i\pi m x_n} \right|^2 \right)^{1/2} .$$

Or in terms of the discrepancy function:

$$F_N^2(\omega) = 2\pi^2 \int_0^1 \int_0^1 \Delta^2([\alpha, \beta), N, \omega) d\alpha d\beta.$$

The crucial point of the concept of discrepancy is that the notion of uniform distribution can be covered by it.

Theorem

A sequence $\omega = (x_n)_{n\geq 1}$ is uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty} D_N(\omega) = 0.$$

Low discrepancy sequences

There are several examples of one dimensional sequences with

$$ND_N(\omega) = \mathcal{O}(\log N).$$

It is therefore convenient to introduce the term **low discrepancy sequence** for sequences with this property. Moreover low discrepancy sequences can be compared by computing this constant

$$\limsup_{N\to\infty} \frac{ND_N(\omega)}{\log N}.$$

Van der Corput Sequence

- Base $b \geq 2$, integers n
- b-adic representation of $n = \sum_{j=0}^{\infty} a_j(n) b^j$

Van der Corput Sequence in Base b:

$S_b = (\phi_b(n))_{n \geq 0}$ with

$$\phi_b(n) = \sum_{j=0}^{\infty} \frac{a_j(n)}{b^{j+1}} \in [0,1)$$

- $\phi_b \ldots$ radical inverse function
- For $n = (a_{m-1}, \ldots, a_1, a_0)_b$ we have $\phi_b(n) = (0.a_0 \ldots a_{m-1})_b$
- Compare $\frac{n}{b^m} = (0.a_{m-1} \ldots a_0)_b$

Halton Sequence and Hammersley Point Set

s-dimensional generalization: choose s pairwise coprime bases b_i.

Halton Sequence in Bases $b_1, \ldots, b_s$:

$H = (H(n))_{n \geq 0}$ with

$$H(n) = (\phi_{b_1}(n), \ldots, \phi_{b_s}(n)) \in [0,1)^s$$

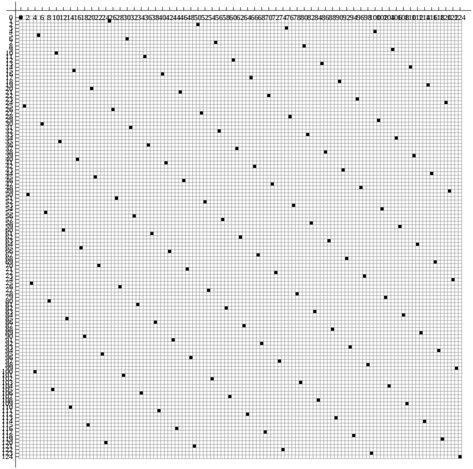
The following point set of N points is closely connected:

Hammersley Point Set in Bases $b_1, \ldots, b_{s-1}$:

$H_N = (H_N(n))_{n=0}^{N-1}$ with

$$H_N(n) = \left(\frac{n}{N}, \phi_{b_1}(n), \ldots, \phi_{b_{s-1}}(n)\right) \in [0,1)^s$$

Van der Corput Sequence in Base 5



Generalized Van der Corput Sequence

- $b \geq 2$, $\Sigma = (\sigma_j)_{j \geq 0}$ sequence of permutations of $\{0, 1, \ldots, b-1\}$
- b-adic representation of $n = \sum_{j=0}^{\infty} a_j(n) b^j$

Generalized Van der Corput Sequence S_b^Σ in Base b

$$S_b^\Sigma(n) = \sum_{j=0}^{\infty} \frac{\sigma_j(a_j(n))}{b^{j+1}}$$

- For $(\sigma_j) = (\sigma)$ constant: write $S_b^\Sigma = S_b^\sigma$
- Original van der Corput sequence for the identical permutation

Generalized 2-Dimensional Hammersley Point Set $\mathcal{H}_{b,N}^\Sigma$ in Base b

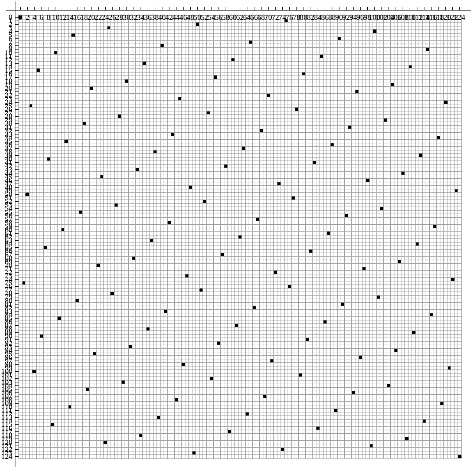
$$\mathcal{H}_{b,N}^\Sigma = \left\{ \left(S_b^\Sigma(n), \frac{n}{N} \right) \,:\, 0 \leq n < N \right\}$$

Example

Let $b = 5$,
$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 & 4 \end{pmatrix}$.

n	$\dots 5^2 5^1 5^0$	x_n
0	000	$\frac{\sigma(0)}{5} = 0$
1	001	$\frac{\sigma(1)}{5} = \frac{3}{5}$
2	002	$\frac{\sigma(2)}{5} = \frac{2}{5}$
3	003	$\frac{\sigma(3)}{5} = \frac{1}{5}$
4	004	$\frac{\sigma(4)}{5} = \frac{4}{5}$
5	010	$\frac{\sigma(0)}{5} + \frac{\sigma(1)}{25} = \frac{3}{25}$
6	011	$\frac{\sigma(1)}{5} + \frac{\sigma(1)}{25} = \frac{3}{5} + \frac{3}{25}$
7	012	$\frac{\sigma(2)}{5} + \frac{\sigma(1)}{25} = \frac{2}{5} + \frac{3}{25}$
8	013	$\frac{\sigma(3)}{5} + \frac{\sigma(1)}{25} = \frac{1}{5} + \frac{3}{25}$
9	014	$\frac{\sigma(4)}{5} + \frac{\sigma(1)}{25} = \frac{4}{5} + \frac{3}{25}$
$\vdots$	$\vdots$	$\vdots$

Generalized Van der Corput Sequence in Base 5



Contributions of Faure

- Detailed study of generalized van der Corput sequences
- Proved explicit formulae for various discrepancy measures for generalized van der Corput sequences
- Developed a technique to compute according constants and therefore to classify permutations
- Found good permutations ($\rightarrow$ recent improvements)

Basic Tools

- Observations are based on system of basic $\varphi^{\sigma}_{b,h}$-functions
- Combinations of these functions suffice to prove formulae for different discrepancy measures
- These functions can even be used to study more dimensional sequences and point sets

Analysis of the irregularities of distribution I

For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b)$. For $h \in \{0, 1, \ldots, b-1\}$ and $x \in \left[\frac{k-1}{b}, \frac{k}{b}\right)$, where $k \in \{1, \ldots, b\}$ we define

Definition

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A([0, \frac{h}{b}), k, \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([\frac{h}{b}, 1), k, \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where for a sequence $\omega = (x_n)_{n \geq 1}$ we denote by $A(J, k, \omega)$ the number of indices $1 \leq n \leq k$ such that $x_n \in J$.

Analysis of the irregularities of distribution II

In the main theorems the following classes of functions based on the basic $\varphi_{b,h}^{\sigma}$ appear:

Definition

$$\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} \left(\varphi_{b,h}^{\sigma}\right)^r \ ,$$

$$\chi_b^{\sigma} := \frac{1}{2} \sum_{h \neq h'} (\varphi_{b,h'}^{\sigma} - \varphi_{b,h}^{\sigma})^2 \ ,$$

$$\psi_b^{\sigma} := \sup_{0 \leq h < h' < b} \left| \varphi_{b,h'}^{\sigma} - \varphi_{b,h}^{\sigma} \right| \ .$$

(See Pausinger for some properties and for a graph of a χ-function)

Illustration

Example with $b = 3$ and $\sigma = \left(\begin{smallmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{smallmatrix}\right)$

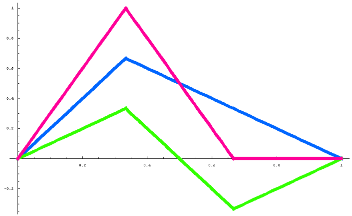


Figure: $\varphi_{3,1}^{\sigma}$ (blue), $\varphi_{3,2}^{\sigma}$ (green), $\varphi_{3}^{\sigma,(1)}$ (red)

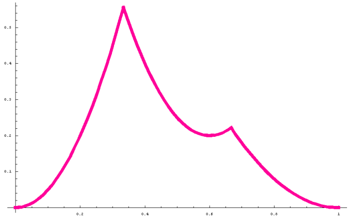


Figure: $\varphi_{3}^{\sigma,(2)}$

Discrepancy: explicit formula and asymptotic behaviour

Faure:

Let N be an integer with $N \geq 1$, then

$$ND_N(S_b^\sigma) = \sum_{j=1}^{\infty} \psi_b^\sigma\left(\frac{N}{b^j}\right) .$$

Let

$$\alpha_{b,\sigma} := \inf_{n \geq 1} \sup_{x \in \mathbb{R}} \left(\frac{1}{n} \sum_{j=1}^{n} \psi_b^\sigma\left(\frac{x}{b^j}\right) \right),$$

then

$$s(S_b^\sigma) = \limsup_{N \to \infty} \frac{N \cdot D_N(S_b^\sigma)}{\log N} = \frac{\alpha_{b,\sigma}}{\log b}.$$

Ψ-Function in Base 5

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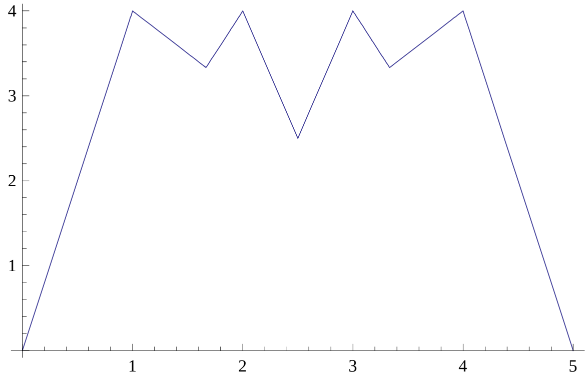

Ψ-Function in Base 5

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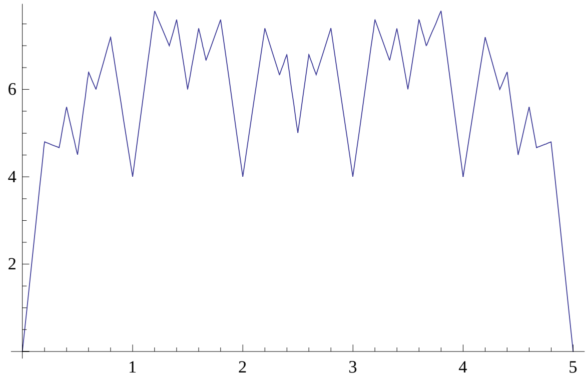

Ψ-Function in Base 5

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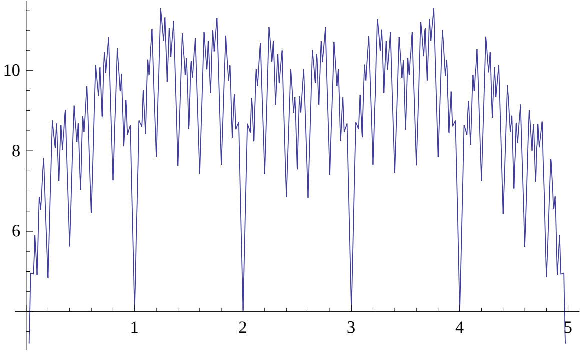

Ψ-Function in Base 5

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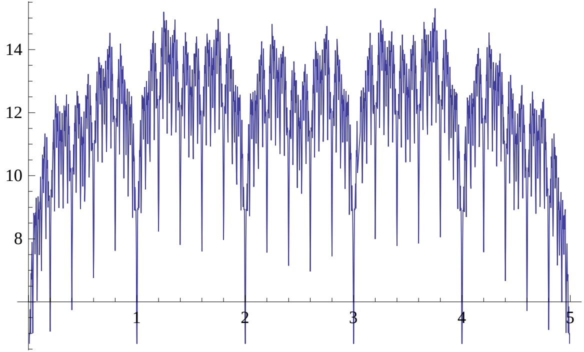

Diaphony: explicit formula

Faure:

Let N be an integer with $N \geq 1$, then

$$N^2 F_N^2(S_b^\sigma) = 4\pi^2 \sum_{j=1}^{\infty} \chi_b^\sigma(Nb^{-j}))/b^2 \ .$$

(See Pausinger for an asymptotic behaviour like before and for lower bounds)

Results for discrepancy and diaphony

$$s(S_b^\sigma) = \limsup_{N\to\infty} \frac{N \cdot D_N(S_b^\sigma)}{\log N}, \qquad f(S_b^\sigma) = \limsup_{N\to\infty} \frac{N \cdot F_N^2(S_b^\sigma)}{\log N}$$

Discrepancy:

- Faure (1992): permutation σ_{36} with $s(S_{36}^{\sigma_{36}}) = 0.366\ldots$
- Pirsic/S (2008): slight improvements in base 36
- Polt (2008): $s(S_{60}^{\sigma_{60}}) = 0.360\ldots$
- Ostromoukhov (2009): $s(S_{84}^{\sigma_{84}}) = 0.353\ldots$

Diaphony:

- Chaix/Faure (1993): $f(S_{19}^{\sigma_{19}}) = 1.315\ldots$
- Pausinger/S (2010): $f(S_{57}^{\sigma_{57}}) = 1.137\ldots$

Results for L_2 and L_p Discrepancy

- Faure/Pillichshammer (2008): existence of permutations such that the L_p discrepancy is of best possible order
  Explicit permutations for the L_2 discrepancy
- Faure/Pillichshammer/Pirsic/S (2009): exact formula for the L_2 discrepancy for special permutations
  $\Longrightarrow$ 2-dimensional finite point sets with the lowest value of L_2 discrepancy known $(0.179\ldots)$
  Recently improved to $0.176\ldots$ by Bilyk et al. (symmetrized Fibonacci sets)
- Faure/Pillichshammer (2009): new results for the L_2 discrepancy of 2-dimensional digitally shifted Hammersley point sets in base b