On the Construction of Quadrature Formulas for SDEs

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Introduction

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Computational task: constructive approximation of

- the distribution $\mu = P_X$ on the path space $\mathcal{X}$,
- the marginal distribution $\mu = P_{X_1}$ on the state space $\mathcal{X}$

by probability measures

$$\widehat{\mu} = \sum_{i=1}^{n} c_i \cdot \delta_{x_i}$$

with finite support $\{x_1, \ldots, x_n\} \subset \mathcal{X}$.

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with finite support $\{x_1, \dots, x_n\} \subset \mathcal{X}$.

Note: $\widehat{\mu}$ yields a quadrature formula

$$\int_{\mathcal{X}} f \, d\widehat{\mu} = \sum_{i=1}^{n} c_i \cdot f(x_i) \quad \text{for} \quad \int_{\mathcal{X}} f \, d\mu.$$

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Challenges:

- distribution μ is given only implicitly by x_0, a, b.
- $\dim(\mathcal{X}) = \infty$ for the path space $\mathcal{X}$.

OUTLINE

I. Algorithms, Error and Cost
II. Quadrature Formulas on the Path Space
III. Quadrature Formulas on the State Space
IV. Final Remarks

Joint project with

M. Scheutzow (Berlin)
S. Dereich, R. Schottstedt (Marburg),
F. Heidenreich, A. Neuenkirch, K. Ritter (Kaiserslautern),
T. M-G, L. Yaroslavtseva (Passau).

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I. Algorithms, Error and Cost

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Put

$$\mathcal{M}(\mathcal{X}) = \Big\{\nu\colon \nu \text{ Borel prob. measure on } \mathcal{X}, \forall\, s \geq 1 : \int_{\mathcal{X}} \|x\|^s \, d\nu(x) < \infty\Big\}$$

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Approximate $\mu\colon \mathcal{H} \to \mathcal{M}(\mathcal{X})$ with

$$\mu(x_0, a, b) = \begin{cases} P_X & \text{for a)}, \\ P_{X_1} & \text{for b)}. \end{cases}$$

Deterministic algorithms

$$\widehat{\mu}\colon \mathcal{H} \to \{\nu \in \mathcal{M}(\mathcal{X})\colon\ |\operatorname{supp}(\nu)| < \infty\},$$

based on finitely many eval's of $a, b, a', b', \ldots$.

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Cost and Error

$$\operatorname{cost}(\widehat{\mu}) = \sup_{(x_0,a,b)\in\mathcal{H}} (\#\text{eval's of } a, b, a', b', ... + \#\text{arithmetical operations})$$

$$e(\widehat{\mu}) = \sup_{(x_0,a,b)\in\mathcal{H}} \rho(\mu(x_0, a, b), \widehat{\mu}(x_0, a, b))$$

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Quantization numbers w.r.t. ρ. For $\nu \in \mathcal{M}(\mathcal{X})$

$$q_n(\nu) = \inf\{\rho(\nu, \widehat{\nu})\colon\ \widehat{\nu} \in \mathcal{M}(\mathcal{X}), |\operatorname{supp}(\widehat{\nu})| \le n\}.$$

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Clearly,

$$e_N \ge \sup_{(x_0,a,b)\in\mathcal{H}} q_N(\mu(x_0, a, b)).$$

II. Quadrature Formulas on the Path Space

Class of SDEs: $(x_0, a, b) \in \mathcal{H} = [-L, L] \times H \times H$, where

$$H = \{h \in C^2(\mathbb{R}) : |h(0)|, |h'|, |h''| \leq K\}$$

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Metric: Wasserstein metric $\rho^{(s)}$ of order $s \geq 1$ on $\mathcal{M}(\mathcal{X})$

$$\rho^{(s)}(\nu, \widehat{\nu}) = \inf_{\xi} \Big(\int_{\mathcal{X} \times \mathcal{X}} \|x - y\|^s \, d\xi(x, y) \Big)^{1/s}$$

with inf over all Borel prob. measures ξ on $\mathcal{X} \times \mathcal{X}$ with marginals $\nu, \widehat{\nu}$.

Theorem (Dereich 2008). For every $(x_0, a, b) \in \mathcal{H}$

$$q_n(\mu(x_0, a, b)) \approx \kappa(x_0, a, b, \mathcal{X}, s) \cdot (\log n)^{-1/2}$$

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Remark

(i) Construction uses Brownian bridge quantization and yields $\widehat{\mu}_n$ with

$$|\operatorname{supp}(\widehat{\mu}_n)| \le n, \quad \operatorname{cost}(\widehat{\mu}_n) \le c \cdot n, \quad e(\widehat{\mu}_n) \le c \cdot (\log n)^{-1/2}.$$

Moreover, if $\mathcal{X} = L_p([0,1])$ and $s = p$ or if $\mathcal{X} = C([0,1])$ then

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(ii) Alternative approaches to constructive quantization

- ODE-based, using rough paths theory, see *Luschgy, Pagès (2006)*, *Pagès, Sellami (2010)*,
- using series expansions for X, see *Luschgy, Pagès (2008)*.

Sketch of Construction for $p = s = 2$.

Idea: Approximate $X = X(x_0, a, b)$ by $\widehat{X} = \widehat{X}^1 + \widehat{X}^2$, where

$\widehat{X}^1 =$ piecewise linear interpolation of Milstein scheme $\widehat{X}_{t_0}, \dots, \widehat{X}_{t_m}$ at times $t_\ell = \ell/m$, (*coarse level*)

$$\widehat{X}_t^2 = \sum_{\ell=1}^{m} b(\widehat{X}_{t_{\ell-1}}) \cdot B_t^{(\ell)} \cdot 1_{[t_{\ell-1}, t_\ell]}(t) \qquad (\textit{local refinement})$$

with Brownian bridges

$$B_t^{(\ell)} = W_t - W_{t_{\ell-1}} - (t - t_{\ell-1}) \cdot m \cdot (W_{t_\ell} - W_{t_{\ell-1}}), \qquad t \in [t_{\ell-1}, t_\ell].$$

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Note: $(\widehat{X}_{t_0}, \ldots, \widehat{X}_{t_m}), B^{(1)}, \ldots, B^{(m)}$ are independent.

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Construct approximations

$$\widehat{\mu}^1 = \sum_{i=1}^{M_1} w_i \cdot \delta_{\mathbf{x}_i} \quad \text{ of } P_{(\widehat{X}_{t_0}, \dots, \widehat{X}_{t_m})},$$

$$\widehat{\mu}_{\mathbf{x}_i}^2 = \sum_{j=1}^{M_2} w_{i,j} \cdot \delta_{f_{i,j}} \quad \text{ of } P_{\sum_{\ell=1}^{m} b(\mathbf{x}_{i,\ell-1}) \cdot B^{(\ell)} \cdot 1_{[t_{\ell-1}, t_\ell]}}, \quad i = 1, \dots, M_1,$$

with $\mathbf{x}_i \in \mathbb{R}^{m+1}$ and functions $f_{i,j} \in L_2([0,1])$.

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with $\mathbf{x}_i \in \mathbb{R}^{m+1}$ and functions $f_{i,j} \in L_2([0,1])$. Approximate $\mu(x_0, a, b) = P_X$ by

$$\widehat{\mu}(x_0, a, b) = \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} w_i \cdot w_{i,j} \cdot \delta_{\mathbf{x}_i + f_{i,j}}$$

Ingredients

- Quantization of $\mathrm{N}(0,1)$: $\nu_n \in \mathcal{M}(\mathbb{R})$ with $|\,\mathrm{supp}(\nu_n)| = n$ and

$$\rho^{(2)}(\mathrm{N}(0,1), \nu_n) \le c \cdot n^{-1}.$$

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Adjust parameters, e.g., $m = n = (\log N)^\alpha$ with $\alpha \in (1/2, 1)$.

Example: square-root diffusion

$$dX_t = \alpha(\kappa - X_t)\,dt + \beta\sqrt{X_t}\,dW_t.$$

with $\alpha = 1.0$, $\kappa = 0.02$, $\beta = 0.2$, $x_0 = 0.04$.

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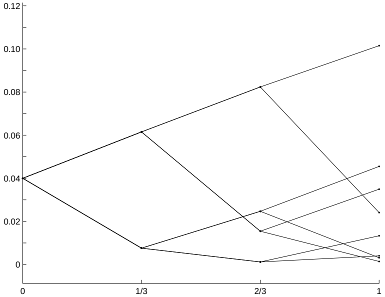
Coarse level via quantization of Milstein scheme with step-size $1/3$:
8 polygons with uniform weight $1/8$.



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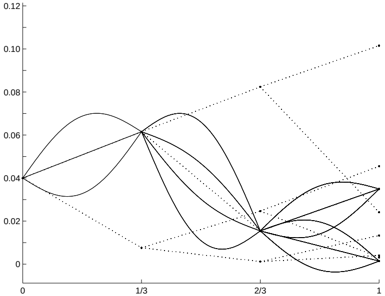
Local refinement via quantization of Brownian bridges based on Karhunen-Loève expansion with basis functions

$$g_k(t) = \sqrt{2}\cdot\sin(k\pi\cdot t), \quad k \in \mathbb{N}.$$



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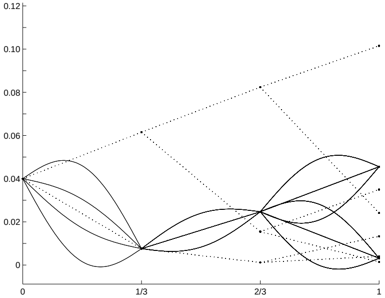

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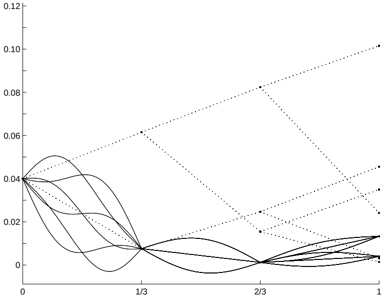

III. Quadrature Formulas on the State Space

Class of SDEs: $(x_0, a, b) \in \mathcal{H} = [-L, L] \times H \times H^\varepsilon$, where

$$H = \{h \in C^4(\mathbb{R})\colon\ |h(0)|, |h^{(j)}| \le K,\ j = 1, \dots, 4\},$$
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Metric: for $\nu, \widehat{\nu} \in \mathcal{M}(\mathbb{R})$

$$\rho_F(\nu, \widehat{\nu}) = \sup_{f \in F} \left| \int f\, d\nu - \int f\, d\widehat{\nu} \right|$$

where

$$F = \{f \in C^4(\mathbb{R}) : |f^{(j)}(z)| \le M \cdot (1 + |z|^\beta),\ j = 1, \ldots, 4\}$$

with $M, \beta \ge 0$.

Theorem (M-G, Ritter, Yaroslavtseva 2011). For every $\delta > 0$

$$e_N \le \begin{cases} c \cdot N^{-2/3+\delta} & \text{if } \varepsilon > 0, \\ c \cdot N^{-1/2+\delta} & \text{if } \varepsilon = 0 \end{cases}$$

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Remark. Related results and alternative approaches

- Weighted integration on $\mathbb{R}^d$: *Wasilkowski, Woźniakowski (2000, 2001)*. For $\mu(x_0, a, b)$ with Lebesgue density satisfying decay conditions,

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- Nonlinear integration problems for SDEs: *Petras, Ritter (2006)*. For a semi-metric $\rho_{\{f\}}$ with fixed f, fixed x_0, $b = 1$, varying $a \in H_{r,d}$,
$$e_N \geq c \cdot N^{-r/d}.$$

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Remark. Related results and alternative approaches

- Quadrature on the Wiener space: *Kusuoka (2001, 2004)*, *Lyons, Victoir (2004)*, *Ninomiya, Victoir (2008)*, *Litterer, Lyons (2010)*, ...
- Finite difference methods via Feynman-Kac representation

Sketch of Construction for $\varepsilon = 1$.

Fix $(x_0, a, b) \in \mathcal{H}$ and $\delta > 0$.

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Leads to **Markov chain** $(Y_k^{x_0})_{k \in \mathbb{N}_0}$ with

- state space $S = G \cup \{x_0\}$, where

$$G = \{ i \cdot m^{-1/2} : \; i = -\lceil m^{1/2+\delta} \rceil, \ldots, \lceil m^{1/2+\delta} \rceil \},$$

- initial value x_0,
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Approximation

$$\widehat{\mu}(x_0, a, b) = \mathbb{P}_{Y_m^{x_0}} = \sum_{z\in S} p^{(m)}_{x_0,z} \cdot \delta_z.$$

Fix $y \in S$. Euler step

$$X^E = \underbrace{y + a(y) \cdot m^{-1}}_{e} + b(y) \cdot m^{-1/2} \cdot V, \quad V \sim \mathrm{N}(0,1).$$

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Construction of probabilities $(p_{y,z})_{z \in S}$ such that $\#\{z : p_{y,z} \neq 0\}$ is small and

$$\left|\mathbb{E}\big(X^E - e\big)^j - \mathbb{E}\big(Y_1^y - e\big)^j\right| \leq c \cdot (1 + |y|^c) \cdot m^{-2}$$

for all $j \in \mathbb{N}$.

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Approach, essentially,

$$\begin{aligned} \text{gridpoints:} \quad & \lfloor e \rfloor_G, \lfloor e \rfloor_G \pm \big\lceil |b(y)| \big\rceil \cdot m^{-1/2}, \\ & \lceil e \rceil_G, \lceil e \rceil_G \pm \big\lceil |b(y)| \big\rceil \cdot m^{-1/2} \\ \text{weights:} \quad & \text{solve} \\ & \mathbb{E}(X^E - e)^j = \mathbb{E}(Y_1^y - e)^j, \quad j = 1, 2, 3. \end{aligned}$$

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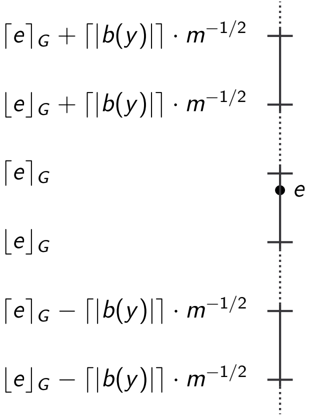
y



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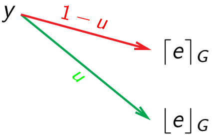

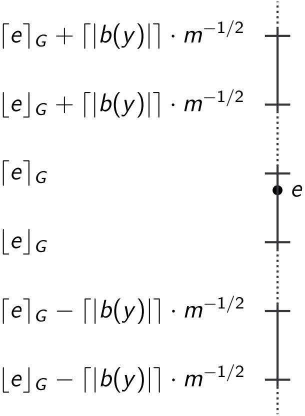

with

$$u = m^{1/2} \cdot (\lceil e \rceil_G - e) \quad .$$

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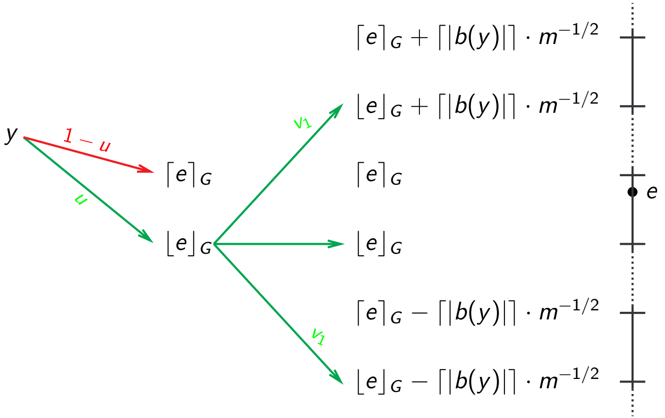

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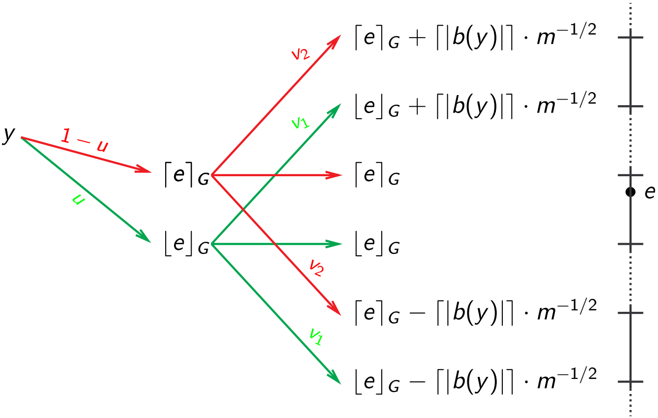

with

$$u = m^{1/2} \cdot (\lceil e \rceil_G - e), \; v_1 = \frac{3b^2(y) + u^2 - 1}{6\lceil |b(y)| \rceil^2}, \; v_2 = \frac{3b^2(y) + u^2 - 2u}{6\lceil |b(y)| \rceil^2}.$$

Cost and Error

Recall

$$\widehat{\mu}(x_0, a, b) = \sum_{z \in S} p^{(m)}_{x_0,z} \cdot \delta_z.$$

We have

	eval's of a, b	arithm. op.
support points, S		$\leq c \cdot m^{1/2+\delta}$
transition probabilities, $(p_{y,z})_{y,z \in S}$	$\leq c \cdot m^{1/2+\delta}$	$\leq c \cdot m^{1/2+\delta}$
weights, $(p^{(m)}_{x_0,z})_{z \in S}$		$\leq c \cdot m^{3/2+\delta}$

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$$cost(\widehat{\mu}) \leq c \cdot m^{3/2+\delta}.$$

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Hence,

$$cost(\widehat{\mu}) \leq c \cdot m^{3/2+\delta}.$$

Furthermore,

$$e(\widehat{\mu}) \leq c \cdot m^{-1}.$$

IV. Final Remarks

- path space:
  d-dim. systems of SDEs need quantization of Lévy-areas
- state space:
  gap between lower bound $N^{-r/d}$ and upper bound $N^{-(r-2)/(d+2)}$
- better results via approximation by general measures?
- what can be achieved with randomized algorithms?