Randomized approximation and solution of elliptic PDEs

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1. Notation

$1 \le p \le \infty$, $d \in \mathbb{N} = \{1, 2, \dots\}$, $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $Q \subset \mathbb{R}^d$ a bounded Lipschitz domain (i.e., locally Lipschitz boundary)

Sobolev space

$$W_p^r(Q) = \{f \in L_p(Q) : \; D^\alpha f \in L_p(Q), \; |\alpha| \le r\}$$

$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| := \sum_{j=1}^d \alpha_j \le r$,
$D^\alpha f$ generalized partial derivative, i.e., in the sense of distributions: $D^\alpha f \in \mathcal{D}'(Q)$

$$(g, D^\alpha f) = (-1)^{|\alpha|}(D^\alpha g, f) \quad (g \in \mathcal{D}(Q))$$

norm

$$\|f\|_{W_p^r(Q)} = \left(\sum_{|\alpha| \le r} \|D^\alpha f\|_{L_p(Q)}^p \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{W_\infty^r(Q)} = \max_{|\alpha| \le r} \|D^\alpha f\|_{L_\infty(Q)}.$$

Note:

$$r = 0 \quad W_p^0(Q) = L_p(Q)$$

Approximation problem

$1 \le p, q \le \infty,\ r, s \in \mathbb{N}_0,$

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right)$$

Approximate

$$J : W_p^r(Q) \to W_q^s(Q)$$

deterministic algorithms $\mathcal{A}_n^{\text{det}}$

$$A : W_p^r(Q) \to W_q^s(Q),$$

$$A(f) = \sum_{i=1}^{n} f(x_i)\psi_i \quad x_i \in Q,\ \psi_i \in W_q^s(Q)$$

these are **linear algorithms**

other classes of algorithms:

nonlinear nonadaptive algorithms

$$A(f) = \varphi(f(x_1), \ldots, f(x_n))$$
$$(x_i \in Q, \quad \varphi : \mathbb{R}^n \to W_q^s(Q))$$

nonlinear adaptive algorithms

$$\begin{aligned} x_2 &= x_2(f(x_1)) \\ x_3 &= x_3(f(x_1), f(x_2)) \\ \cdots & \quad \cdots \\ x_n &= x_n(f(x_1), \ldots, f(x_{n-1})) \end{aligned}$$

So far all algorithm classes were based on **standard information** $\Lambda^{\text{st}} = \{\delta_t : t \in Q\}$, that is, function values

linear information $\Lambda^{\text{lin}} = W_p^r(Q)^*$ that is, arbitrary linear continuous functionals

$$A(f) = \sum_{i=1}^{n} (f, v_i)\psi_i \quad v_i \in W_p^r(Q)^*, \ \psi_i \in W_q^s(Q)$$

and analogously for the other classes

error:

$$e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \\ = \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - A(f)\|_{W_q^s(Q)}$$

deterministic n-th minimal error:
(linear sampling numbers)

$$e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \\ = \inf_{A \in \mathcal{A}_n^{\text{det}}} e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

randomized algorithms $\mathcal{A}_n^{\text{ran}}$:

$(\Omega, \Sigma, \mathbb{P})$ probability space,

$$(A_\omega)_{\omega\in\Omega}, \quad A_\omega : W_p^r(Q) \to W_q^s(Q)$$

$$A_\omega(f) = \sum_{i=1}^{n} f(x_{i,\omega})\psi_{i,\omega}$$

$x_{i,\omega} \in Q$, $\psi_{i,\omega} \in W_q^s(Q)$ $(\omega \in \Omega)$,

error:

$$e(J, (A_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) = \sup_{f\in\mathcal{B}_{W_p^r(Q)}} \mathbb{E}\,\|Jf - A_\omega(f)\|_{W_q^s(Q)}$$

randomized n-th minimal error:

(randomized linear sampling numbers)

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) = \inf_{(A_\omega)\in\mathcal{A}_n^{\text{ran}}} e(J, (A_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \leq e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

e_n^{det} : $\quad \Omega = \{\omega_0\}$

embedding condition

$W_p^r(Q)$ is embedded into $C(Q)$ iff

$$\left.\begin{array}{lll} & p = 1 \quad \text{and} \quad r/d \geq 1 \\ \text{or} & \\ & 1 < p \leq \infty \quad \text{and} \quad r/d > 1/p \end{array}\right\} \quad (1)$$

2. Embedding $J : W_p^r(Q) \to W_q^s(Q)$ with $s \geq 0$

Theorem 1. *(many authors)*
Let $r, s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$, *with*

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right),$$

let Q *be a bounded Lipschitz domain. Then in the deterministic setting, if the embedding condition (1) holds,*

$$e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \asymp n^{-\frac{r-s}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+},$$

if the embedding condition does not hold, then

$$e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)} \cap C(Q), W_q^s(Q)) \asymp 1.$$

In the randomized setting

$$e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \asymp n^{-\frac{r-s}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+}.$$

independently of the embedding condition.

Case $s = 0$:

deterministic, $D = [0, 1]^d$, embedding condition:
classical approximation theory, Sard

deterministic, D bounded Lipschitz domain, embedding condition:
Novak, Triebel, 2006

deterministic, D bounded Lipschitz domain, no embedding condition:
H. 2007, **Novak, Woźniakowski**

randomized, $D = [0, 1]^d$, embedding condition:
Wasilkowski 1989, **Novak** 1988, **Mathé** 1991

randomized, D bounded Lipschitz domain, no embedding condition: **H.** 2009

Case $s > 0$:

deterministic, $D = [0, 1]^d$, embedding condition:

classical approximation theory,

Vybiral 2007 (various function spaces)

deterministic, D bounded Lipschitz domain, embedding condition, but nonlinear sampling numbers:

Triebel 2005

Case $s > 0$ (continued):

deterministic, D bounded Lipschitz domain, embedding condition, linear sampling numbers: **H.** 2009

solving Problem 18 of Novak, Woźniakowski (Tractability of Multivariate Problems, Volume 1)

randomized, D bounded Lipschitz domain, no embedding condition: **H.** 2009

3. Embedding into spaces with negative smoothness

$J : W_p^r(Q) \to W_q^{-s}(Q)$

How is $W_q^{-s}(Q)$ defined?

Consider $W_{q^*}^s(Q) \quad (\frac{1}{q} + \frac{1}{q^*} = 1)$

$\widetilde{W}_{q^*}^s(Q)$ – closure of functions $g \in W_{q^*}^s(Q)$ with

$$\text{supp}(g) \subset Q$$

(Q open, so compactly contained), in the norm of $W_{q^*}^s(Q)$.

By duality we define (for $1 < q \leq \infty$)

$$W_q^{-s}(Q) := \widetilde{W}_{q^*}^s(Q)^*$$

that is $f \in \mathcal{D}'(Q)$ is in $W_q^{-s}(Q)$ iff

$$|(g, f)| \leq c\|g\|_{W_{q^*}^s(Q)}$$

For example, if $s/d > 1/q \Rightarrow \widetilde{W}_{q^*}^s(Q) \subset C(Q) \Rightarrow$ $\delta_t \in W_q^{-s}(Q)$,
if $s/d > 1/q + 1/d \Rightarrow \widetilde{W}_{q^*}^s(Q) \subset C^1(Q) \Rightarrow$ $\delta_t' \in W_q^{-s}(Q)$, etc.

Motivation/application: weak elliptic problem

$m \in \mathbb{N}$, bilinear form a on $W_2^m(Q)$,

$$a(u,v) = \sum_{|\alpha|,|\beta|\le m} \int_Q a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x)\,dx,$$

$a_{\alpha\beta} \in C(Q)$

assume a is $\widetilde{W}_2^m(Q)$-elliptic

$$\begin{aligned} |a(u,v)| &\le c_1 \|u\|_{W_2^m(Q)} \|v\|_{W_2^m(Q)} \\ a(u,u) &\ge c_2 \|u\|^2_{W_2^m(Q)} \end{aligned}$$

$(u, v \in \widetilde{W}_2^m(Q))$.

Consequently, $a(u,\cdot\,) \in \widetilde{W}_2^m(Q)^* = W_2^{-m}(Q)$

$$L_a : u \to a(u,\cdot\,), \quad L_a \in \mathscr{L}(\widetilde{W}_2^m(Q), W_2^{-m}(Q))$$

$$\|L_a : \widetilde{W}_2^m(Q) \to W_2^{-m}(Q)\| \le c_1$$

$$
\begin{aligned}
(L_a u, u) &\geq c_2 \|u\|^2_{W_2^m(Q)} \\
\|L_a u\|_{W_2^{-m}(Q)} &\geq c_2 \|u\|_{W_2^m(Q)}
\end{aligned}
$$

(and L_a is onto) $\Rightarrow$

$$
\|L_a^{-1} : W_2^{-m}(Q) \to \widetilde{W}_2^m(Q)\| \leq c_2^{-1}.
$$

So L_a^{-1} is an isomorphism.

weak elliptic problem associated with a:

Given $f \in W_2^{-m}(Q)$, find $u \in \widetilde{W}_2^m(Q)$ such that for all $v \in \widetilde{W}_2^m(Q)$

$$\begin{aligned} a(u,v) &= f(v) \\ (\Leftrightarrow\ L_a u &= f) \end{aligned}$$

The problem has a unique solution

$$L_a^{-1} f \in \widetilde{W}_2^m(Q)$$

associated differential operator:

$$\mathscr{L} u = \sum_{|\alpha|,|\beta| \le m} D^\beta (a_{\alpha\beta}\, D^\alpha u)$$

associated differential equation (under further smoothness assumptions on $a_{\alpha,\beta}$, f and ∂Q): given f, find u with

$$\begin{aligned} \mathscr{L} u &= f \quad \text{on} \quad Q \\ D_n^\gamma u &= 0 \quad \text{on} \quad \partial Q \quad (0 \le \gamma \le m-1) \end{aligned}$$

Let $r \in \mathbb{N}_0$.
Our task: solve the weak problem for $f \in W_2^r(Q)$, i.e., find $u \in \widetilde{W}_2^m(Q)$ such that

$$a(u,v) = f(v) \quad (v \in \widetilde{W}_2^m(Q))$$

solution operator

$$S^{\text{ell}} = L_a^{-1} J : W_2^r(Q) \xrightarrow{J} W_2^{-m}(Q) \xrightarrow{L_a^{-1}} \widetilde{W}_2^m(Q)$$

Corollary 1.

$$\begin{aligned} & e_n^{\text{ran}}(S^{\text{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \\ \asymp \quad & e_n^{\text{ran}}(J, \mathcal{B}_{W_2^r(Q)}, W_2^{-m}(Q)), \end{aligned}$$

and analogously for e_n^{det}.

Approximation of the Embedding

$J : W_p^r(Q) \to W_q^{-s}(Q)$

Deterministic case:

Theorem 2. *Let* $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $1 \le p, q \le \infty$,

$$\frac{r+s}{d} > \frac{1}{p} - \frac{1}{q}.$$

1. (Novak, Triebel, 2006, Vybiral, 2007) Assume that $W_p^r(Q)$ *is embedded into* $C(Q)$*. Then*

$$e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp n^{-\gamma_1}$$

where

$$\gamma_1 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d}\right).$$

2. (H., 2009) If $W_p^r(Q)$ *is not embedded into* $C(Q)$*, then*

$$e_n^{\mathrm{det}}(J : \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp 1.$$

Randomized case:

Theorem 3. *(H., 2009) Let* $r \in \mathbb{N}_0$*,* $s \in \mathbb{N}$*,* $1 \leq p, q \leq \infty$ *and*

$$\frac{r+s}{d} > \frac{1}{p} - \frac{1}{q}.$$

Then

$$e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp_{log} \; n^{-\gamma_2}$$

where

$$\gamma_2 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right)$$

and $\bar{p} = \min(p, 2)$*.*

solved Problem 25 of Novak, Woźniakowski (Tractability of Multivariate Problems, Volume 1)

For the **weak elliptic problem** we conclude

Corollary 2. *Let* $r \in \mathbb{N}_0$, $m \in \mathbb{N}$. *Then*

$$e_n^{\text{det}}(S^{\text{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \asymp n^{-\frac{r}{d}}.$$

Moreover, if $\frac{m}{d} \neq \frac{1}{2}$, *then*

$$e_n^{\text{ran}}(S^{\text{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \asymp n^{-\frac{r}{d} - \min\left(\frac{m}{d}, \frac{1}{2}\right)}$$

while for $\frac{m}{d} = \frac{1}{2}$

$$\begin{aligned} c_3 n^{-\frac{r}{d} - \frac{1}{2}} &\leq e_n^{\text{ran}}(S^{\text{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \\ &\leq c_4 n^{-\frac{r}{d} - \frac{1}{2}} \log n \end{aligned}$$

4. Simultaneous approximation in different norms

Theorem 4. *(H., 2011) Let* $r \in \mathbb{N}_0$*,* $s, s_1 \in \mathbb{N}_0$*,* $1 \le p, q, q_1 \le \infty$ *be such that* $W_p^r(Q) \subset W_{s_1}^{q_1}(Q)$ *and* $W_p^r(Q) \subset W_q^{-s}(Q)$*. Then there exists a linear randomized algorithm* $(U_\omega)_{\omega\in\Omega}$*, such that*

$$e(J, (U_\omega), \mathcal{B}_{W_p^r(Q)}, W_{q_1}^{s_1}(Q)) \;\asymp\; n^{-\gamma_1}.$$

with

$$\gamma_1 = \frac{r - s_1}{d} + \left(\frac{1}{p} - \frac{1}{q_1}\right)_+$$

and

$$e(J, (U_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp_{log} \; n^{-\gamma_2}$$

where

$$\gamma_2 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right)$$

and $\bar{p} = \min(p, 2)$*.*

(U_ω) has the form

$$U_\omega(f) = \sum_{i=1}^{n} f(x_{i,\omega})\psi_{i,\omega} \quad (f \in W_p^r(Q))$$

with $x_{i,\omega} \in Q$, $\psi_{i,\omega} \in W_{q_1}^{s_1}(Q) \cap W_q^{-s}(Q)$ $(\omega \in \Omega)$

Corollary 3. *Let*

$$S = S_0 J : W_p^r(Q) \xrightarrow{J} W_q^{-s}(Q) \xrightarrow{S_0} Z$$

where Z *is a Banach space and* $S_0 \in \mathscr{L}(W_q^{-s}(Q), Z)$. *Then*

$$\begin{aligned} & e_n^{\text{ran,st}}(S, \mathcal{B}_{W_r^p(Q)}, Z) \\ & \quad \le \; c\, e_n^{\text{det,lin}}(S, \mathcal{B}_{W_r^p(Q)}, Z) \\ & \qquad + c\, e(J, (U_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)). \end{aligned}$$

Proof: Let $\delta > 0$ and let

$$A(f) \;=\; \sum_{i=1}^{n} (f, v_i) z_i \quad (v_i \in W_p^r(Q)^*,\ z_i \in Z)$$

be such that

$$e(S, A, \mathcal{B}_{W_r^p(Q)}, Z) \leq e_n^{\text{det,lin}}(S, \mathcal{B}_{W_r^p(Q)}, Z) + \delta$$

$$\sup_{f \in \mathcal{B}_{W_r^p(Q)}} \|Sf - A(f)\| \leq e_n^{\text{det,lin}}(S, \mathcal{B}_{W_r^p(Q)}, Z) + \delta$$

Let $\tilde{A}_\omega = A \circ U_\omega$, where U_ω satisfies Theorem 4 with $q_1 = p$ and $s_1 = r$, that is,

$$\mathbb{E}\, \|U_\omega(f)\|_{W_r^p(Q)} \leq c \|f\|_{W_r^p(Q)}.$$

For $f \in \mathcal{B}_{W_r^p(Q)}$ we have

$$
\begin{aligned}
&\mathbb{E}\,\|Sf - \tilde{A}_\omega(f)\| \\
&\quad = \mathbb{E}\,\|S_0 Jf - A(U_\omega(f))\| \\
&\quad \le \mathbb{E}\,\|S_0 Jf - S_0 U_\omega(f)\| \\
&\qquad\quad + \mathbb{E}\,\|\underbrace{S_0 U_\omega(f)}_{= SU_\omega(f)} - A(U_\omega(f))\| \\
&\quad \le \|S_0\| e(J, (U_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \\
&\qquad + \Big(e_n^{\mathrm{det,lin}}(S, \mathcal{B}_{W_r^p(Q)}, Z) + \delta\Big)\,\mathbb{E}\,\|U_\omega(f)\|_{W_r^p(Q)}
\end{aligned}
$$

Corollary 4. *Let*

$$S^{\text{ell}} : W_2^r(Q) \xrightarrow{J} W_2^{-m}(Q) \xrightarrow{L_a^{-1}} \widetilde{W}_2^m(Q)$$

be as defined above and let A be any deterministic linear algorithm for S^{ell} based on continuous linear functionals Λ^{lin}. Then

$$\begin{aligned} & e(S, (A \circ U_\omega), \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \\ \leq \;& c\, e(S, A, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \\ & + c\, n^{-\frac{r}{d} - \min\left(\frac{m}{d}, \frac{1}{2}\right)} (\log n)^\sigma. \end{aligned}$$

with

$$\sigma = \begin{cases} 0 & \textit{if} \quad \frac{m}{d} \neq \frac{1}{2} \\ 1 & \textit{if} \quad \frac{m}{d} = \frac{1}{2}. \end{cases}$$

Moreover, the arithmetic cost of $(A \circ U_\omega)$ is that of A plus the cost of computing

$$(\psi_{i,\omega}, v_j) \quad (i, j = 1, \ldots, n).$$

Proof

We apply algorithm A not to $f \in W_2^r(Q)$, but to

$$U_\omega(f) = \sum_{i=1}^{n} f(x_{i,\omega})\psi_{i,\omega} \in W_2^r(Q)$$

So

$$\begin{aligned}(A \circ U_\omega)(f) &= \sum_{i,j=1}^{n} f(x_{i,\omega})(\psi_{i,\omega}, v_j) z_j \\ & \qquad (z_j \in \widetilde{W}_2^m(Q))\end{aligned}$$

Note that

$$\begin{aligned}& e_n^{\text{det,lin}}(S^{\text{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \\ \asymp \;& e_n^{\text{det,lin}}(J, \mathcal{B}_{W_2^r(Q)}, W_2^{-m}(Q)) \asymp n^{-\frac{r+m}{d}}.\end{aligned}$$

5. The algorithm

We assume for simplicity $1 < p = q < \infty$, $Q = [0,1]^d$,

$$J = J_2 J_1 : W_p^r(Q) \xrightarrow{J_1} L_p(Q) \xrightarrow{J_2} W_p^{-s}(Q)$$

duality

$$\begin{aligned} J_2 &: L_p(Q) \to W_p^{-s}(Q) = \widetilde{W}_{p^*}^s(Q)^* \\ J_0 &: \widetilde{W}_{p^*}^s(Q) \to L_{p^*}(Q) \qquad J_2 = J_0^* \end{aligned}$$

$$\|J_0 - P_k : \widetilde{W}_{p^*}^s(Q) \to L_{p^*}(Q)\| \le c2^{-sk}$$

$$P_k h = \sum_{i=1}^{n_k} (h, \tilde{h}_{ki}) \tilde{g}_{ki},$$

$n_k \asymp 2^{dk}$, $\quad \tilde{h}_{ki} \in \widetilde{W}_{p^*}^s(Q)^*$,

$\tilde{g}_{ki} \in L_{p^*}(Q)$ $\quad$ almost disjoint supports

$$\|J_0^* - P_k^* : L_p(Q) \to W_p^{-s}(Q)\| \le c2^{-sk}$$

$$P_k^* f = \sum_{i=1}^{n_k} (f, \tilde{g}_{ki}) \tilde{h}_{ki},$$

idea: use simultaneous Monte Carlo integration for the approximation of the weighted integrals

$$(f, \tilde{g}_{ki}) = \int_Q f(x) \tilde{g}_{ki}(x) dx$$

Does not give the optimal rate!

Multilevel splitting:

$$P_k = \sum_{l=0}^{k} (P_l - P_{l-1}), \qquad P_{-1} = 0$$

$$\|P_l - P_{l-1} : \widetilde{W}_{p^*}^s(Q) \to L_{p^*}(Q)\| \le c2^{-sk}$$

$$(P_l - P_{l-1})h = \sum_{i=1}^{n_l} (h, h_{li}) g_{li}, \quad h_{li} \in \widetilde{W}^s_{p^*}(Q)^*,$$

$g_{ki} \in L_{p^*}(Q)$ linearly independent, almost disjoint supports,

$\Rightarrow$ approximate

$$(P_l^* - P_{l-1}^*)f = \sum_{i=1}^{n_l} (f, g_{li}) h_{li},$$

More general task: Z Banach space,
$T : Z \to L_{p^*}(Q)$ bounded linear operator, with

$$Tz = \sum_{i=1}^{n} (z, z_i^*) g_i$$

and for $u = p^*, \infty$

$$c_1 n^{-1/u} \|(\lambda_i)\|_{l_u^n} \le \left\| \sum_{i=1}^{n} \lambda_i g_i \right\|_{L_u(Q)} \le c_2 n^{-1/u} \|(\lambda_i)\|_{l_u^n}$$

approximate $T^*|_{L_p(Q)} : L_p(Q) \to Z^*$

ξ_i uniformly distributed on Q_i

$$\gamma_i(f) := |Q_i| g_i(\xi_i) f(\xi_i)$$

$$\mathbb{E}\,\gamma_i(f) = \int_{Q_i} g_i(x) f(x) dx = (g_i, f)$$

$$T^* f = \sum_{i=1}^{n} (f, g_i) z_i^*$$

$$\approx \quad A_{n,N}(f) = \sum_{i=1}^{n} \left(\frac{1}{N} \sum_{j=1}^{N} \gamma_{ij}(f) \right) z_i^*$$

$\gamma_{ij}(f)$ independent copies of $\gamma_i(f)$.

Further assumptions:

$$\operatorname{supp} g_i \subseteq Q_i, \qquad |Q_i| \le c_3 n^{-1},$$

$$\max_{x \in Q} |\{i : x \in Q_i\}| \le c_4 \quad (Q_i \text{ almost disjoint}).$$

Proposition 1. *Let* $\bar{p} = \min(2, p)$. *Then*

$$\left(\mathbb{E}\, \|T^* f - A_{n,N}(f)\|_{Z^*}^{\bar{p}} \right)^{1/\bar{p}} \le c \|T\| N^{-1+1/\bar{p}} \|f\|_{L_p(Q)}.$$

Idea of proof:

$$\left\|\sum_{i=1}^{n}\lambda_i z_i^*\right\|_{Z^*} = \sup_{z\in\mathcal{B}_Z}\left|\sum_{i=1}^{n}\lambda_i(z,z_i^*)\right|$$

$$\leq \sup_{z\in\mathcal{B}_Z}\|(z,z_i^*)\|_{l_{p^*}^n}\|(\lambda_i)\|_{l_p^n}$$

$$\leq c_1^{-1}n^{1/p^*}\sup_{z\in\mathcal{B}_Z}\|\sum_{i=1}^{n}(z,z_i^*)g_i\|_{L_{p^*}(Q)}\|(\lambda_i)\|_{l_p^n}$$

$$\leq c_1^{-1}n^{1/p^*}\|T\|\,\|(\lambda_i)\|_{l_p^n}$$

l_p^n is uniformly of type $\min(2, p)$.

For simplicity let $p \geq 2$.

$$\mathbb{E}\, \|T^* f - A_{n,N}(f)\|_{Z^*}^2$$

$$\begin{aligned}
&= \mathbb{E} \left\| \sum_{i=1}^{n} \left((f, g_i) - \frac{1}{N} \sum_{j=1}^{N} \gamma_{ij}(f) \right) z_i^* \right\|_{Z^*}^2 \\
&\leq cn^{2/p^*} \|T\|^2 \, \mathbb{E} \left\| \left((f, g_i) - \frac{1}{N} \sum_{j=1}^{N} \gamma_{ij}(f) \right)_{i=1}^{n} \right\|_{l_p^n}^2 \\
&\leq cn^{2/p^*} \|T\|^2 N^{-2} \sum_{j=1}^{N} \mathbb{E} \left\| \Big((f, g_i) - \gamma_{ij}(f) \Big)_{i=1}^{n} \right\|_{l_p^n}^2 \\
&\leq cn^{2/p^*} \|T\|^2 N^{-1} \, \mathbb{E} \, \|(\gamma_i(f))\|_{l_p^n}^2 \\
&\leq cn^{2/p^*} \|T\|^2 N^{-1} \left(\mathbb{E} \, \|(\gamma_i(f))\|_{l_p^n}^p \right)^{2/p}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{E}\,|\gamma_i(f)|^p &= \sum_{i=1}^{n} |Q_i|^{p-1} \int_{Q_i} |g_i(x) f(x)|^p dx \\
&\leq \sum_{i=1}^{n} |Q_i|^{p-1} \|g_i(x)\|_\infty \int_{Q_i} |f(x)|^p dx \\
&\leq c n^{-p+1} \sum_{i=1}^{n} \int_{Q_i} |f(x)|^p dx \\
&\leq c n^{-p+1} \int_{Q} |f(x)|^p dx
\end{aligned}$$

$\square$

$$
\begin{aligned}
J_2 f &\approx A_k f \\
&= \sum_{l=0}^{k} \sum_{i=1}^{n_l} \left(\frac{1}{N_l} \sum_{j=1}^{N_l} |Q_{li}| g_{li}(\xi_{lij}) f(\xi_{lij}) \right) h_{li}
\end{aligned}
$$

$$
\text{Total error} \leq c \sum_{l=0}^{k} 2^{-sl} N_l^{-1/2}
$$

$$
\text{cost} \leq \sum_{l=0}^{k} 2^{dl} N_l
$$

$$
Jf \approx U_k f + A_k(f - U_k f),
$$

where

$$
(\,\mathbb{E}\,)\, \|f - U_k f\|_{L_p(Q)} \leq c\, 2^{-rk} \|f\|_{W_p^r(Q)}
$$