The Randomized Setting for Linear Multivariate Problems defined over L_2

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Based on

Tractability of Multivariate Problems
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Erich Novak + H.W

Multivariate Problems

cont. + linear + nonzero $S_d \quad : \quad F_d \to G_d$ Hilbert spaces

$$S_d(f) \quad \sim \quad A_n(f)$$

Worst Case Setting : $A_n(f) = \phi_n(L_1(f), \dots, L_n(f))$

Randomized Setting : $A_n(f,\omega) = \phi_{n(\omega),\omega}(L_{1,\omega}(f), \dots, L_{n(\omega),\omega}(f))$

ω a random element, $\mathbb{E}_\omega n(\omega) \le n$

Standard Information $\Lambda^{\rm std}$: $L_j(f) = f(x_j),\ L_{j,\omega}(f) = f(x_{j,\omega})$

Linear Information $\Lambda^{\rm all}$: $L_j, L_{j,\omega}$ – linear functionals

see Hinrichs, Novak + W [2011]

Settings

Errors :

Worst : $$e^{\mathrm{wor}}(A_n) = \sup_{\|f\|_{F_d}\le 1} \|S_d(f) - A_n(f)\|$$

Randomized : $$e^{\mathrm{ran}}(A_n) = \sup_{\|f\|_{F_d}\le 1} \left(\mathbb{E}_\omega \|S_d(f) - A_n(f,\omega)\|^2\right)^{1/2}$$

Information Complexity : **sett** $\in \{$**wor, ran**$\}$, $\Lambda \in \{\Lambda^{\mathrm{std}}, \Lambda^{\mathrm{all}}\}$

$$n^{\mathrm{sett}}(\varepsilon, S_d, \Lambda) = \min\{n \mid \exists\, A_n \text{ with } L_j \in \Lambda \ \ e^{\mathrm{sett}}(A_n) \le \varepsilon\, e^{\mathrm{sett}}(0)\}$$

RAN $\sim$ WOR for $\Lambda^{\rm all}$

Known: (Novak[88], Wasilkowski [88], . . .)

$$\tfrac{1}{4}\, n^{\rm wor}(2\varepsilon, S_d, \Lambda^{\rm all}) \le n^{\rm ran}(\varepsilon, S_d, \Lambda^{\rm all}) \le n^{\rm wor}(\varepsilon, S_d, \Lambda^{\rm all})$$

For compact S_d :

$$n^{\rm wor}(\varepsilon, S_d, \Lambda^{\rm all}) = \min\{\, n \;\mid\; \lambda_{n+1} \le \varepsilon^2\,\lambda_1\} \;<\; \infty \quad \forall\ \varepsilon > 0,$$

where

$$W_d\eta_j = \lambda_j\eta_j \quad \text{for} \quad W_d = S_d^*S_d \;\; \text{and} \;\; \lambda_1 \ge \lambda_2 \ge \cdots \ge 0,\;\; \lambda_n \to 0$$

Optimal algorithm in the Worst Case: $n = n^{\rm wor}(\varepsilon, S_d, \Lambda^{\rm all})$,

$$A_n(f) = \sum_{j=1}^{n} \langle f, \eta_j \rangle_{F_d}\, S_d\eta_j$$

Class $\Lambda^{\rm std}$ of Function Values

Main question:

What is the power of $\Lambda^{\rm std}$
as compared to the power of $\Lambda^{\rm all}$?

or when

$$n^{\rm ran}(\varepsilon, S_d, \Lambda^{\rm std}) \sim n^{\rm ran}(\varepsilon, S_d, \Lambda^{\rm all})$$

If so then

RAN for $\Lambda^{\rm std}$ $\sim$ Ran for $\Lambda^{\rm all}$ $\sim$ WOR for $\Lambda^{\rm all}$

This is the case for, see Wasilkowski + W [2006],

$$S_d f = {\rm APP}_d f := f \ \in \ G_d = L_{2,\rho}(D_d)$$

$$F_d = L_{2,\rho_d}(D_d)$$

From now on we assume

$$F_d = L_{2,\rho_d}(D_d)$$

with

$$\|f\|_{F_d} = \left(\int_{D_d} f^2(x)\, \rho_d(x)\, \mathrm{d}x \right)^{1/2}$$

ρ_d prob. density function

Lower Bounds

For all (nonzero) S_d

$$n^{\text{ran}}(\varepsilon, S_d, \Lambda^{\text{std}}) \ \geq \ \frac{1}{4}\ \varepsilon^{-2}$$

independently of how small is

$$n^{\text{ran}}(\varepsilon, S_d, \Lambda^{\text{all}})$$

Lower and Upper Bounds

If

$$k \;:=\; \sup_d \;\dim(\,S_d(F_d))\;<\;\infty$$

then

$$n^{\rm ran}(\varepsilon, S_d, \Lambda^{\rm std}) = a(\varepsilon,d)\,\varepsilon^{-2} \qquad \textbf{with} \qquad a(\varepsilon,d) \in [1/4, k+1]$$

Hence

$$n^{\rm ran}(\varepsilon, S_d, \Lambda^{\rm std}) \;\sim\; \left(\frac{1}{\varepsilon}\right)^2$$

All finite dimensional linear multivariate problems are strongly polynomially tractable with the same exponent $= 2$

Arbitrary cont.+linear +nonzero S_d

For sett $\in \{$wor, ran$\}$

$$e_n^{\mathrm{sett}}(S_d,\Lambda) = \inf_{A_n \text{ with } L_j \in \Lambda} e^{\mathrm{sett}}(A_n)$$

Then

$$e_n^{\mathrm{wor}}(S_d,\Lambda^{\mathrm{all}})^2 = \lambda_{n+1}$$

$$\frac{\lambda_1}{4\,n} \le e_n^{\mathrm{ran}}(S_d,\Lambda^{\mathrm{std}})^2 \le \lambda_{m+1} + \frac{\sum_{j=1}^m \lambda_j}{n} \qquad \forall\, m$$

λ_j **ordered eigenvalues of** $W_d = S_d^* S_d$

Weak Tractability

$S = \{S_d\}$ **is weakly tractable iff**

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\rm sett}(\varepsilon, S_d, \Lambda)}{\varepsilon^{-1}+d} = 0$$

Then the following statements are equivalent

- **Weak Trac. for Rand. and** $\Lambda^{\rm std}$
- **Weak Trac. for Rand. and** $\Lambda^{\rm all}$
- **Weak Trac. for Worst and** $\Lambda^{\rm all}$

Polynomial Tractability

$S = \{S_d\}$ **is (strongly) pol. tractable iff** $\exists\ C, p, (q = 0)\, q \geq 0$

$$n^{\text{sett}}(\varepsilon, S_d, \Lambda) \ \leq C\, \varepsilon^{-p}\, d^{\,q} \qquad \forall\ \varepsilon \in (0,1),\ d = 1, 2, \dots$$

Then the following statements are equivalent

- **(St.) Pol. Tract. for Rand. and** Λ^{std}
- **(St.) Pol. Tract. for Rand. and** Λ^{all}
- **(St.) Pol. Tract. for Worst and** Λ^{all}

Polynomial Tractability

But the exponents are sometimes different

$$n^{\text{sett}}(\varepsilon, S_d, \Lambda) \le C\, \varepsilon^{-p^{\text{sett}}(\Lambda)}\, d^{\,q^{\text{sett}}(\Lambda)} \qquad \forall\ \varepsilon \in (0,1),\ d = 1, 2, \dots$$

Then **for** $p = p^{\text{wor}}(\Lambda^{\text{all}})$ **and** $q = q^{\text{wor}}(\Lambda^{\text{all}})$

$$\begin{aligned} p^{\text{ran}}(\Lambda^{\text{all}}) &= p \\ q^{\text{ran}}(\Lambda^{\text{all}}) &= q \\ p^{\text{ran}}(\Lambda^{\text{std}}) &= \max(2, p) \\ q \le q^{\text{ran}}(\Lambda^{\text{std}}) &\le \max(2q/p, q) \end{aligned}$$

Proof Technique: Lower Bounds

Lemma: If $S_d \to \mathbb{R}$ and

- $f_1, \dots, f_N \in F_d$ with $\|f_j\|_{F_d} = 1$
- f_j's have disjoint support and $S_d f_j \geq \eta > 0$

then

$$e_n^{\mathrm{ran}}(S_d, \Lambda^{\mathrm{std}})^2 \geq \left(1 - \frac{n}{N}\right) \eta$$

based on Novak [88] and Lemma 1 of Volume II, p. 63

For $F_d = L_{2,\rho_d}$

- replace S_d by $\langle f, \phi \rangle_{F_d}$ with $\phi = \eta_1 = \eta_1(S_d)$, $\|\phi\|_{F_d} = 1$
- decompose D_d into $D_{d,j}$ with $\int_{D_{d,j}} \phi^2(t)\,\mathrm{d}t = N^{-1}$
- take $f_j = \sqrt{N}\,\phi\,1_{D_{d,j}}$

Proof Technique: Upper Bounds

Eigenpairs of $W_d = S_d^* S_d$: $\quad W_d \eta_j = \lambda_j \eta_j$

Opt. Alg. in the Worst Case : $\quad A_n(f) = \sum_{j=1}^{n} \langle f, \eta_j \rangle_{F_d} S_d \eta_j$

For $F_d = L_{2,\rho_d}$

$$\langle f, \eta_j \rangle_{F_d} \sim \frac{1}{n} \sum_{\ell=1}^{n} \frac{f(\tau_\ell)\, \eta_j(\tau_\ell)}{u_m(\tau_\ell)},$$

where

$$u_m(t) = \frac{\sum_{j=1}^{m} \lambda_j\, \eta_j^2(t)}{\sum_{j=1}^{m} \lambda_j}$$

$\tau_1, \dots, \tau_n$ iid with respect to $\omega_m = \rho_d\, u_m$.