On the asymptotics of a lower bound for the diaphony of generalized van der Corput sequences in arbitrary base b

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Borovets, August 30th, 2011

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Research supported by the Austrian Science Foundation, project S9609

Outline

- Basic definitions
  - Diaphony
  - Generalized van der Corput sequences
  - Basic tools due to Faure
- Ideas of lower bound
  - Ideas and illustration
  - Our main Theorem
- Asymptotic and numerical results

Generalized van der Corput Sequence

Definition

For a fixed base $b \geq 2$ and a permutation $\sigma \in \mathfrak{S}_b$ the **generalized van der Corput sequence** S_b^σ is defined by

$$S_b^\sigma(n) = \sum_{j=0}^{\infty} \frac{\sigma(a_j(n))}{b^{j+1}},$$

where $\sum_{j=0}^{\infty} a_j(n) b^j$ is the b-adic representation of the integer $n \geq 1$.

Diaphony

Let $X = (x_n)_{n \geq 1}$ be a one-dimensional infinite sequence. The diaphony F of the first N points of X is defined by (Zinterhof, 1976)

$$F(N, X) := \left(2 \cdot \sum_{m=1}^{\infty} \frac{1}{m^2} \left| \frac{1}{N} \sum_{n=1}^{N} \exp^{2i\pi m x_n} \right|^2 \right)^{1/2}.$$

Analysis of the diaphony I

For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b)$. For $h \in \{0, 1, \ldots, b-1\}$ and $x \in \left[\frac{k-1}{b}, \frac{k}{b}\right)$, where $k \in \{1, \ldots, b\}$ we define

Definition

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A([0, \frac{h}{b}); k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([\frac{h}{b}, 1); k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where for a sequence $X = (x_n)_{n \geq 1}$ we denote by $A(I; k; X)$ the number of indices $1 \leq n \leq k$ such that $x_n \in I$.

Analysis of the diaphony II

In 1993 Chaix and Faure introduced a new class of functions based on the basic $\varphi_{b,h}^{\sigma}$:

Definition

$$\chi_b^{\sigma} := \frac{1}{2}\sum_{h \neq h'}(\varphi_{b,h'}^{\sigma} - \varphi_{b,h}^{\sigma})^2.$$

Note that χ_b^{σ} is continuous and piecewise quadratic on the intervals $[\frac{k}{b}, \frac{k+1}{b}]$.

Important Property

We have the following important proposition:

Chaix/Faure, 1993

For each interval $[\frac{k}{b}, \frac{k+1}{b}]$ the parabolic arcs of χ_b^σ are translated versions of the parabola $y = b^2(b^2 - 1)x^2/12$.

Remark

This means that it suffices to know the values of χ_b^σ at the b-adic positions $\frac{k}{b}$ since for every interval $[\frac{k}{b}, \frac{k+1}{b}]$ χ_b^σ is of the form $Ax^2 + Bx + C$ with A only depending on the base b.

Graph of a χ-function

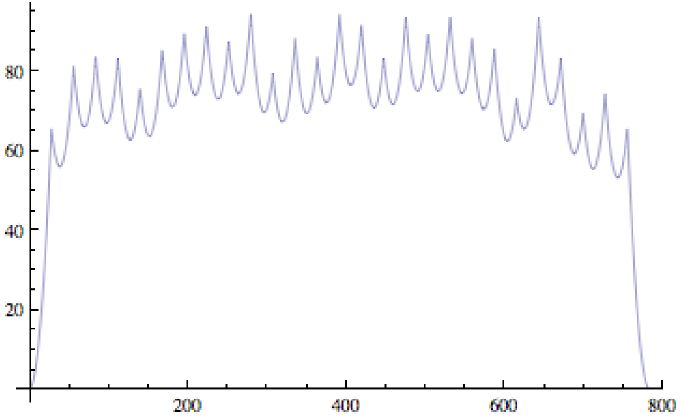

Figure: Graph of a χ-function in base 28

Explicit formula

Chaix/Faure, 1993

Let N be an integer with $N \geq 1$, then

$$F^2(N, S_b^\sigma) = \frac{4\pi^2}{N^2} \sum_{j=1}^{\infty} \chi_b^\sigma(Nb^{-j}))/b^2.$$

Crucial Lemma

In 2005 H. Faure proved the following lemma

Faure, 2005

For any permutation σ in arbitrary base b it holds that

$$\chi_b^{\sigma} \leq \chi_b^{id},$$

where *id* denotes the identical permutation in base b.

Idea of the proof

In the proof of the lemma, sums of the following form are compared for fixed k:

$$\sum_{l(h,h')=d} (\delta^{\sigma}_{h,h'})^2 \text{ and } \sum_{l(h,h')=d} (\delta^{id}_{h,h'})^2, \tag{1}$$

with the condition

$$\sum_{l(h,h')=d} (\delta^{\sigma}_{h,h'}) = \sum_{l(h,h')=d} (\delta^{id}_{h,h'}) = kd, \tag{2}$$

where $d := |h - h'|$ with $0 \leq h, h' \leq b - 1$ and $\delta^{\sigma}_{h,h'} := A([\frac{h}{b}, \frac{h'}{b}); k; Z^{\sigma}_b)$.

Systematic look at these sums

For fixed k we sum over all pairs (h, h') with $h \neq h'$ and $d = |h - h'|$. We can take a more systematic look at these sums and rewrite them as $(b-1) \times b$ matrices, where

- the i in the i-th row denotes b times the length of the interval $[\frac{h}{b}, \frac{h'}{b})$,
- the j in the j-th column stands for the left bound of the interval
- and at position (i, j) we put the number $\delta^{\sigma}_{h,h'}$ of the first k points that actually lie in the interval.

Illustration

For a permutation σ we denote the matrix corresponding to the first k points with $\mathcal{M}_k^\sigma$. Moreover we define $\mathcal{N}(\mathcal{M}_k^\sigma)$ as the sum over the squares of all entries of the matrix.

$$
\begin{array}{rcl}
 & \begin{array}{ccccccc} 0 & \cdot & \cdot & j & \cdot & \cdot & b-1 \end{array} & \\
\begin{array}{r} d=1 \\ \cdot \\ \cdot \\ d=i \\ \cdot \\ \cdot \\ d=b-1 \end{array} &
\begin{pmatrix}
\delta^\sigma_{0,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \delta^\sigma_{b-1,0} \\
\cdot & \cdot & & & & \cdot & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & & \delta^\sigma_{j,j\oplus i} & & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & & & \cdot & \cdot \\
\delta^\sigma_{0,b-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \delta^\sigma_{b-1,b-2}
\end{pmatrix} &
\begin{array}{l} \sum_{row} = k \\ \cdot \\ \cdot \\ \sum_{row} = ik \\ \cdot \\ \cdot \\ \sum_{row} = (b-1)k \end{array}
\end{array}
$$

Illustration

If a new point is added at position j, in other words if we turn from k to $k+1$ points, then the following triangle of 1s is added to the matrix $\mathcal{M}_k^\sigma$ of step k where the 1 in the first row is placed at $\delta_{0,j}^\sigma$:

$$\begin{pmatrix}
 & & & 1 & & & & & \\
 & & . & . & & & & & \\
 & . & & . & & & & & \\
1 & . & . & 1 & & & & & \\
. & & & . & & & & & 1 \\
. & & & . & & & & . & . \\
. & & & . & & & . & & . \\
1 & . & . & 1 & 0 & 1 & . & . & 1
\end{pmatrix}$$

Example

For example for let $b = 6$, $k = 2$. On the left side the first two points are at positions 0, 1, whereas on the right side the points are at positions 0, 3.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 2 \end{pmatrix}$$

In the left matrix we have 10 times 1 and 10 times 2 whereas in the right matrix we have 18 times 1 and only 6 times 2.

Properties of $\mathcal{M}_k^\sigma$

Lemma

For fixed base b and $\sigma \in \mathfrak{S}_b$ and for $1 \le k \le b$ the corresponding matrix $\mathcal{M}_k^\sigma = (m_{i,j})$ has the following properties:

- **(i)** For each row i of $\mathcal{M}_k^\sigma$ we have $\sum_{j=0}^{b-1} m_{i,j} = ik$.
- **(ii)** For each row i of $\mathcal{M}_k^\sigma$ it holds that $|m_{i,j} - m_{i,j+1}| \in \{0,1\}$.
- **(iii)** For each column j of $\mathcal{M}_k^\sigma$ we have $m_{i,j} \le m_{i+1,j}$.
- **(iv)** For each $m_{i,j}$ we have $m_{i,j} \le \min\{i,k\}$.
- **(v)** $\sum_{u=0}^{b} \#(u) = b(b-1)$ and $\sum_{u=0}^{b} \#(u)u = k\frac{(b-1)b}{2}$.

Summary

- To derive a lower bound for any χ_b^σ- function in base b we exploit the first property of these matrices. (See next slides)
- This lower bound can be improved if one takes additionally the special structure of the triangles of 1s into account.
(P. & Schmid, 2011)

Lower Bound

For $kd \equiv \gamma \mod b$, with $\gamma \in \{0, \ldots, b-1\}$, let

$$L_b^{k,d} := (b-\gamma) \cdot \left(\left\lfloor \frac{kd}{b} \right\rfloor \right)^2 + \gamma \cdot \left(\left\lfloor \frac{kd}{b} + 1 \right\rfloor \right)^2 .$$

P. & Schmid , 2011

For arbitrary base b and arbitrary $\sigma \in \mathfrak{S}_b$ and for each k, $1 \leq k \leq b$, it holds that

$$\chi_b^\sigma \left(\frac{k}{b} \right) \geq L_b^k$$

with $L_b^k = \frac{1}{2} \sum_{d=1}^{b-1} \left(L_b^{k,d} - \frac{d^2 k^2}{b} \right)$.

Illustration

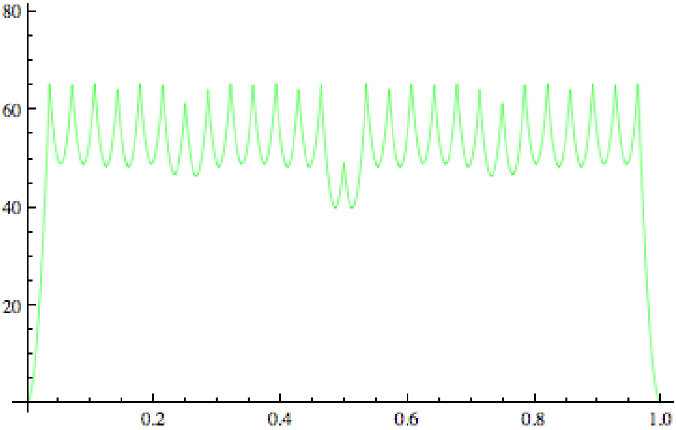

Figure: Lower bound function in base 28.

Illustration

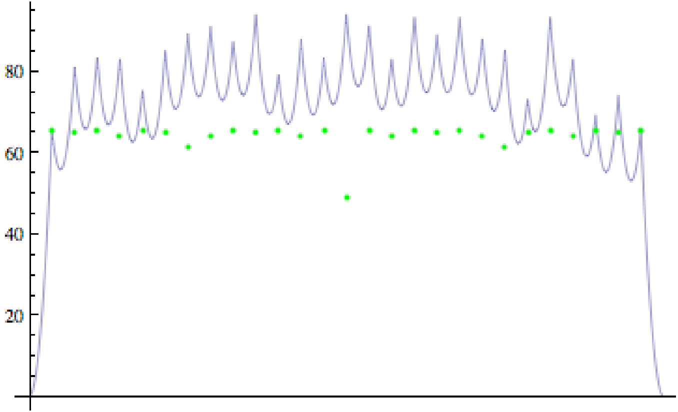

Figure: χ-function in base 28 with values for lower bound (Green).

Important Theorem

Chaix/Faure, 1993

Let

$$\gamma_b^\sigma = \inf_{n \geq 1} \sup_{x \in \mathbb{R}} \left(\sum_{j=1}^{n} \chi_b^\sigma(x/b^j)/n \right),$$

then

$$f(S_b^\sigma) := \limsup_{N \to \infty} \frac{N\, F^2(N, X)}{\log N} = \frac{4\pi^2 \gamma_b^\sigma}{b^2 \log b}.$$

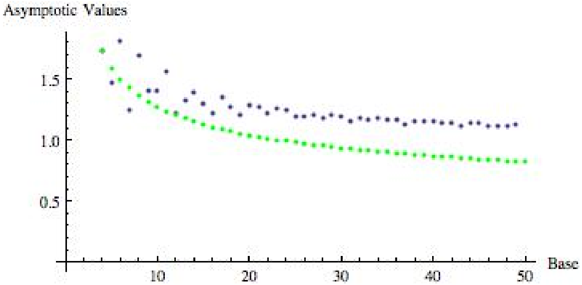

Figure: Asymptotic estimations for diaphony of S_b^σ. Blue: Bounds based on heuristics. Green: Values given by our lower bound.

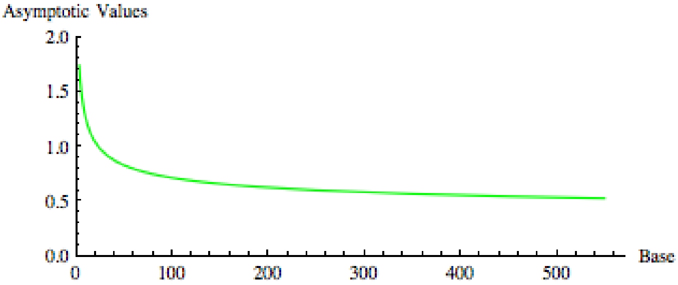

Figure: Asymptotic values given by our lower bound.

Thank you for your attention

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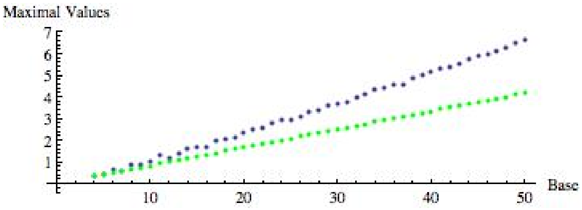

Figure: Smallest possible $\max_{x \in \mathbb{R}} \chi_b^\sigma(x)$ values (blue), lower bound values (green). (Normalized values)

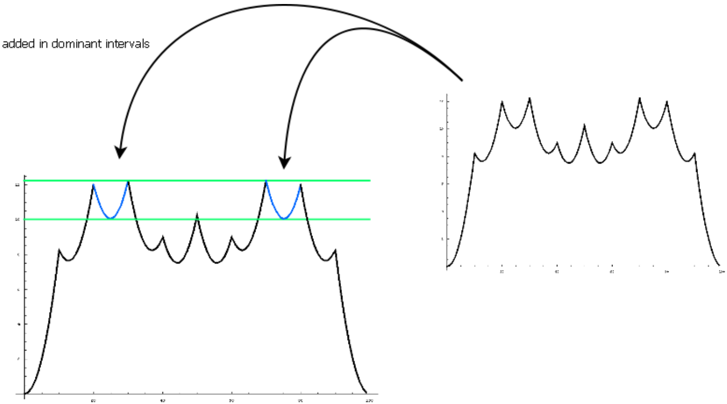

Figure: Idea of asymptotic formula