Multilevel Monte Carlo for Lévy Driven SDEs

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joint work with Steffen Dereich
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Outline

The Problem

SDE

$$\begin{aligned} dY_t &= a(Y_{t-})\, dX_t, \qquad t \in [0,1], \\ Y_0 &= y_0, \end{aligned}$$

with

- a square integrable Lévy process $X = (X_t)_{t\in[0,1]}$,
- a deterministic initial value y_0,
- a Lipschitz continuous function a.

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for functionals $f : D[0,1] \to \mathbb{R}$.

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Example: payoff f of a path dependent option.

Standard Monte Carlo

For approximation $\hat{Y}$ of Y,

$$S(f) = \mathbb{E}[f(Y)] \approx \mathbb{E}[f(\hat{Y})] \approx \underbrace{\frac{1}{n}\sum_{i=1}^{n} f(\hat{Y}_i)}_{=:\hat{S}^{MC}(f)},$$

where $(\hat{Y}_i)_{i=1,\dots,n}$ i.i.d. copies of $\hat{Y}$.

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Error decomposition into weak error and Monte Carlo error

$$\mathbb{E}[|S(f) - \hat{S}^{MC}(f)|^2] = |\underbrace{\mathbb{E}[f(Y) - f(\hat{Y})]}_{=:\mathrm{bias}(f(\hat{Y}))}|^2 + \underbrace{\frac{1}{n}\mathrm{Var}(f(\hat{Y}))}_{=\mathrm{Var}(\hat{S}^{MC}(f))}.$$

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Classical approach: Use approximation $\hat{Y}$ with small bias, e.g. higher-order weak Itô-Taylor schemes, or extrapolation techniques.

Path-independent functions f in the diffusion case

Classical setting:

- Brownian motion as driving process, i.e. $X = W$.
- Compute expectations w.r.t. marginal in $T > 0$, i.e. $S(f) = \mathbb{E}[f(Y(T))]$ for $f : \mathbb{R} \to \mathbb{R}$.

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Relation of error and computational cost: Weak approximation of order β yields

$$\mathbb{E}[|S(f) - \hat{S}^{MC}(f)|^2] \leq \epsilon^2 \quad \text{with computational cost} \quad O(\epsilon^{-2-\frac{1}{\beta}}).$$

Remedy: Only for sufficiently smooth $f \in C^{2(\beta+1)}$.

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Note: Variance reduction via Multilevel algorithm $\hat{S}^{ML}$ yields

$$\mathbb{E}[|S(f) - \hat{S}^{ML}(f)|^2] \leq \epsilon^2 \quad \text{with computational cost} \quad O(\epsilon^{-2}\log(\epsilon)^2).$$

Holds for f Lipschitz.

Multilevel Monte Carlo Algorithms

- See Heinrich [1998], Giles [2006], Creutzig, Dereich, Müller-Gronbach, Ritter [2009], Avikainen [2009], Kloeden, Neuenkirch, Pavani [2009], Hickernell, Müller-Gronbach, Niu, Ritter [2010], Marxen [2010],...
- Talks by Burgos, Giles, Roj, Szpruch, Xia...

Multilevel Monte Carlo Algorithms

Basic idea:

- $Y^{(1)}, Y^{(2)}, \ldots$ strong approximations for the solution Y with increasing accuracy and increasing numerical cost.
- Telescoping sum:

$$\mathbb{E}[f(Y^{(m)})] = \mathbb{E}[\underbrace{f(Y^{(1)})}_{=D^{(1)}}] + \sum_{k=2}^{m} \mathbb{E}[\underbrace{f(Y^{(k)}) - f(Y^{(k-1)})}_{=D^{(k)}}]$$

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- Approximate $\mathbb{E}[D^{(1)}], \ldots, \mathbb{E}[D^{(m)}]$ by independent Monte Carlo methods

$$\hat{S}(f) = \sum_{k=1}^{m} \frac{1}{n_k} \sum_{i=1}^{n_k} D_i^{(k)},$$

for $(D_i^{(k)})_{i=1,\ldots,n_j}$ being i.i.d. copies of $D^{(k)}$, $k = 1, \ldots, m$.

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Note: $(Y^{(k)}, Y^{(k-1)})$ are coupled via X so that the variance of $D^{(k)}$ decreases.

Approximation of the driving Lévy process X

Lévy-Ito-decomposition: X sum of three independent processes.

$$X_t = \sigma W_t + bt + L_t, \qquad t \geq 0.$$

- Brownian motion W,
- drift $b = \mathbb{E}[X_1] \in \mathbb{R}$,
- L_2-martingale of compensated jumps L (jump-intensity measure ν).

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Thus: Cutoff of the small jumps.

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$$X^{(h,\varepsilon)} = (\sigma W_t + bt + L_t^{(h)})_{t \in T^{(h,\varepsilon)}}$$

on random time discretization $T^{(h,\varepsilon)}$ with parameters $h, \varepsilon > 0$

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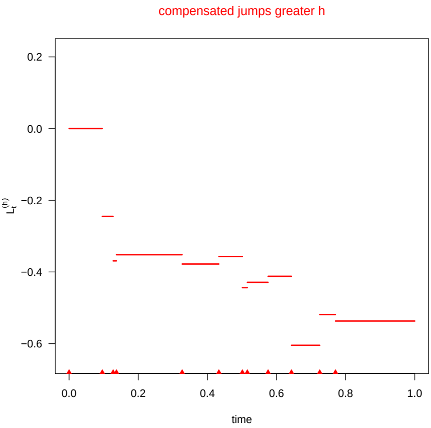
on random time discretization $T^{(h,\varepsilon)}$ with parameters $h, \varepsilon > 0$ such that

- all the jump times of $L^{(h)}$ are included in $T^{(h,\varepsilon)}$,
- step-size is at most ε (for approximation of W).

The coupled approximation

Compound Poisson approximations $(L^{(h)}, L^{(h')})$.



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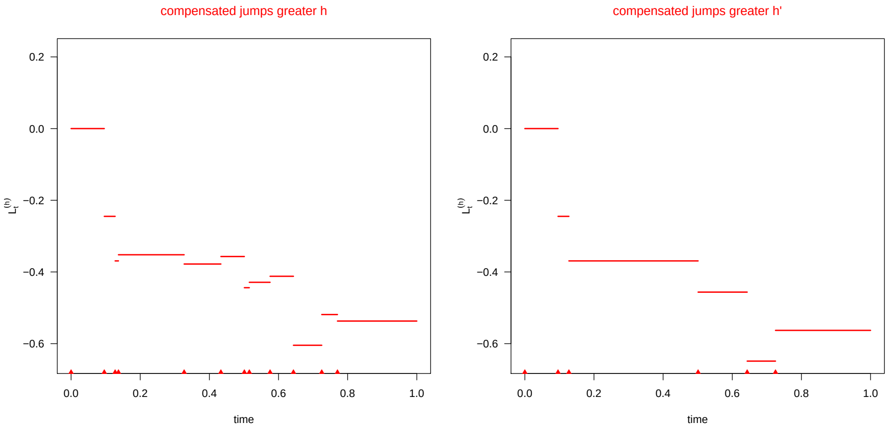
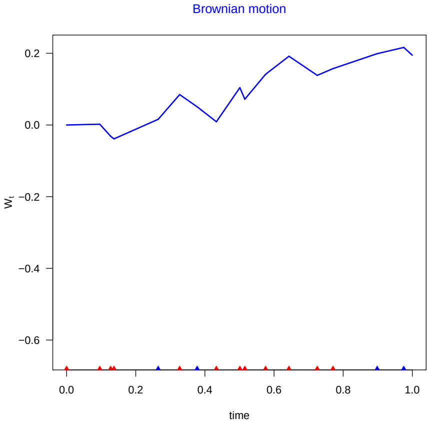


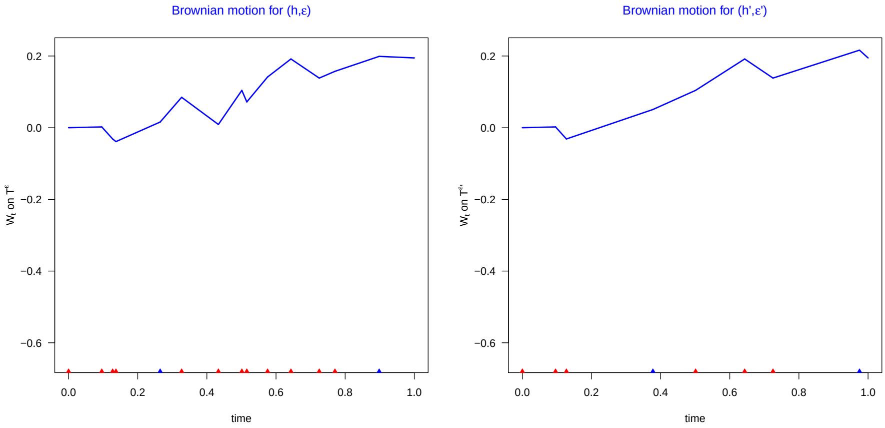
The coupled approximation

Approximation of σW on $T^{(h,\epsilon)} \cup T^{(h',\epsilon')}$.



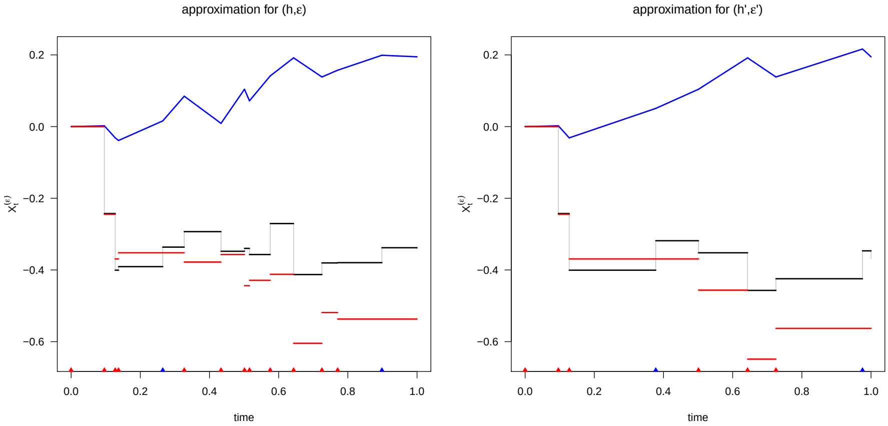
The coupled approximation

Approximation of σW on $T^{(h,\epsilon)}$ and $T^{(h',\epsilon')}$.



The coupled approximation

Piecewise constant approximation of $(X^{(h,\epsilon)}, X^{(h',\epsilon')})$.



Multilevel Euler Algorithm

Remember:

- Telescoping sum

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Parameters of the algorithm $\hat{S}(f)$:

- m (number of levels)
- $h_1 > h_2 > \ldots > h_m$ (approximation of L)
- $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_m$ (approximation of W)
- $n_1, \ldots, n_m$ (number of replications per level)

Result

Error: $$e^2(\hat{S}) = \sup_{f \in \mathrm{Lip}(1)} \mathbb{E}[|S(f) - \hat{S}(f)|^2]$$

Here: $\mathrm{Lip}(1) := \{f : D[0,1] \to \mathbb{R} \mid f\ 1 - \text{Lipschitz w.r.t. supremum norm}\}$

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Theorem (Dereich, H.)

Let

$$\beta := \inf \left\{ p > 0 : \int_{(-1,1)} |x|^p \, \nu(dx) < \infty \right\} \in [0, 2]$$

denote the **Blumenthal-Getoor-index** *of the driving Lévy process. Then, for suitably chosen parameters,* $\mathrm{cost}(\hat{S}_n) \precsim n$ *and*

$$e(\hat{S}_n) \precsim n^{-(\frac{1}{\beta \vee 1} - \frac{1}{2})}.$$

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Remark: Gaussian approximation improves result for $\beta \geq 1$ (see Dereich, 2010).
Order of convergence: $\frac{4-\beta}{6\beta} \wedge \frac{1}{2}$ for $\sigma = 0$ or $\beta \notin [1, 4/3]$,
and $\frac{\beta}{6\beta-4}$ for $\sigma \neq 0$ and $\beta \in [1, 4/3]$.

Remarks

Related results:

- Jacod, Kurtz, Méléard and Protter [2007]
- Marxen [2010]

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Lower bound: Combine results of

- Creutzig, Dereich, Müller-Gronbach and Ritter [2009] on lower bounds in relation to quantization numbers and average Kolmogorov widths, and
- Aurzada and Dereich [2009] on the coding complexity of Lévy processes.

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In the case $X = Y$, for every randomized algorithm $\tilde{S}_n$ with $\mathrm{cost}(\tilde{S}_n) \precsim n$,

$$e(\tilde{S}_n) \succsim n^{-\frac{1}{2}}$$

up to some logarithmic terms.

Example: α-stable processes

Truncated symmetric α-stable processes with $\alpha < 2$ and truncation level $u > 0$

- Lebesgue density of the Lévy measure

$$g_\nu(x) = \frac{c}{|x|^{1+\alpha}} \mathbb{1}_{(-u,u)}(x), \qquad x \in \mathbb{R} \backslash \{0\}.$$

- Blumenthal-Getoor-index $\beta = \alpha$.

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- By the Theorem $e(\hat{S}_n) \precsim \begin{cases} n^{-1/2}, & \alpha < 1, \\ n^{-(1/\alpha - 1/2)}, & \alpha > 1. \end{cases}$

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- By the Theorem $e(\hat{S}_n) \precsim \begin{cases} n^{-1/2}, & \alpha < 1, \\ n^{-(1/\alpha - 1/2)}, & \alpha > 1. \end{cases}$
- In the sequel: X with $u = 1$ and $c = 0.1$, SDE with $a(X_t) = X_t$ and $y_0 = 1$, Lookback option with fixed strike 1

$$f(Y) = \big(\sup_{t\in[0,1]} (Y_t) - 1\big)^+.$$

Adaptive choice of m and $n_1, \ldots, n_m$

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Given $\delta > 0$, determine m and $n_1, \ldots, n_m$ such that

$$\mathbb{E}[|S(f) - \hat{S}(f)|^2] = |\underbrace{\mathbb{E}[f(Y) - f(Y^{(m)})]}_{=\mathrm{bias}(f(Y^{(m)}))}|^2 + \mathrm{Var}(\hat{S}(f)) \leq \delta^2.$$

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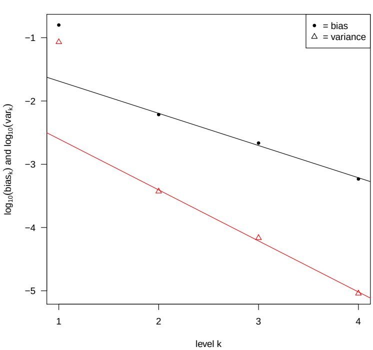
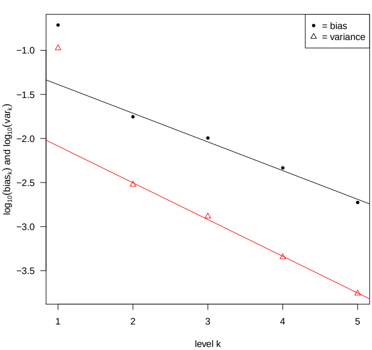
First step: Estimate expectations $\mathbb{E}[D^{(1)}], \ldots, \mathbb{E}[D^{(5)}]$ and variances $\mathrm{Var}(D^{(1)}), \ldots, \mathrm{Var}(D^{(5)})$.

Bias and variance estimation for α=0.5



Bias and variance estimation for α=0.8



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Highest level of accuracy m**:**

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Highest level of accuracy m:

- $\mathbb{E}[D^{(1)}], \dots, \mathbb{E}[D^{(5)}]$ together with extrapolation give control of the $\mathrm{bias}(f(Y^{(k)}))$ for $k \in \mathbb{N}$.
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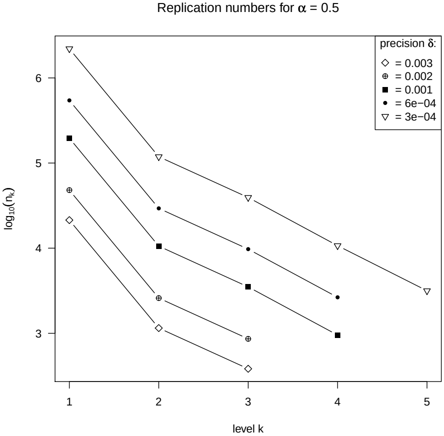
Replication numbers $n_1, \ldots, n_m$:

- $\mathrm{Var}(D^{(1)}), \ldots, \mathrm{Var}(D^{(5)})$ together with extrapolation give control of $\mathrm{Var}(D^{(k)})$ for $k = 1, \ldots, m$.
- Choose $n_1, \ldots, n_m$ with minimal $\mathrm{cost}(\hat{S})$ such that
$$\mathrm{Var}(\hat{S}) = \sum_{k=1}^{m} \frac{\mathrm{Var}(D^{(k)})}{n_k} \leq \frac{\delta^2}{2}.$$

Adaptive choice of m and $n_1, \ldots, n_m$

Expample of $n_1, \ldots, n_m$ with $\alpha = 0.5$.

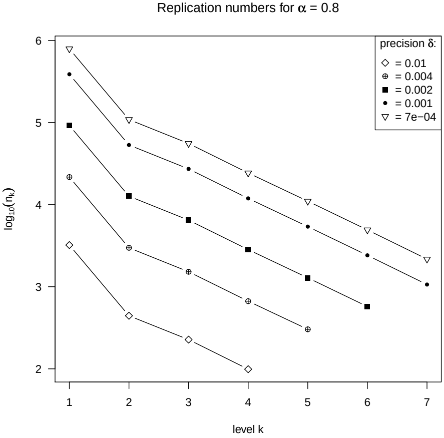
- Precisions $\delta = (0.003, 0.002, 0.001, 0.0006, 0.0003)$.
- Highest levels $m = (3, 3, 4, 4, 5)$.



Adaptive choice of m and $n_1, \ldots, n_m$

Expample of $n_1, \ldots, n_m$ with $\alpha = 0.8$.

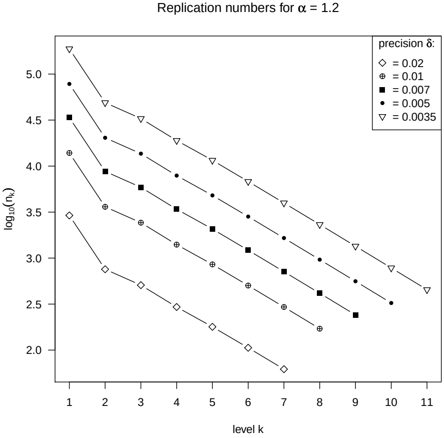
- Precisions $\delta = (0.01, 0.004, 0.002, 0.001, 0.0007)$.
- Highest levels $m = (4, 5, 6, 7, 7)$.



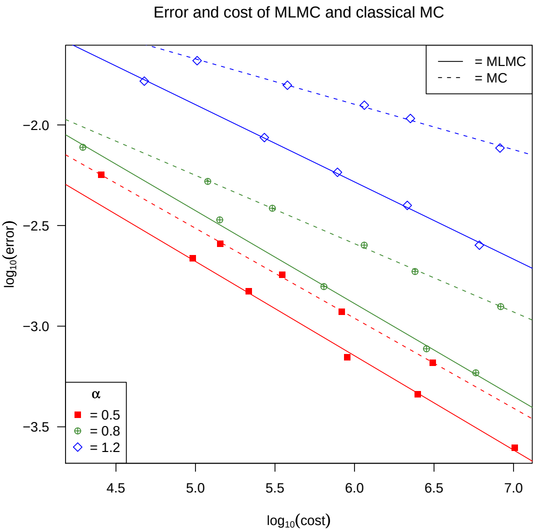
Adaptive choice of m and $n_1, \ldots, n_m$

Expample of $n_1, \ldots, n_m$ with $\alpha = 1.2$.

- Precisions $\delta = (0.02, 0.01, 0.007, 0.005, 0.0035)$.
- Highest levels $m = (7, 8, 9, 10, 11)$.



Empirical relation of error and cost



$\alpha =$	0.5	0.8	1.2
MLMC	0.47	0.46	0.38
MC	0.45	0.34	0.23

Summary and Conclusions

Summary:

- Multilevel algorithm for a wide class of functionals and processes.
- Up to log's asymptotically optimal for Blumenthal-Getoor-index $\beta \leq 1$.
- Heuristics to achieve desired precision $\delta > 0$.
- Numerical results match the analytical results.
- Improvement for $\beta \geq 1$ via Gaussian correction, see Dereich [2010].

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- Up to log's asymptotically optimal for Blumenthal-Getoor-index $\beta \leq 1$.
- Heuristics to achieve desired precision $\delta > 0$.
- Numerical results match the analytical results.
- Improvement for $\beta \geq 1$ via Gaussian correction, see Dereich [2010].

Work to do:

- Further examples of Lévy processes.
- Implementation of the ML-algorithm with Gaussian correction.
- The gap between the upper and lower bound for $\beta > 1$.