Approximating stochastic integrals in L_p

Anni Toivola

University of Jyväskylä, Finland

August 29, 2011

Joint work with Stefan Geiss

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation
- Aim: Observe from the payoff function f “directly” the approximation properties of $f(X_1)$

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation
- Aim: Observe from the payoff function f “directly” the approximation properties of $f(X_1)$
- Simulate $\int_0^t \phi(s, X_s) dX_s$ by a martingale, not computing $\phi(s, X_s)$ too often

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation
- Aim: Observe from the payoff function f “directly” the approximation properties of $f(X_1)$
- Simulate $\int_0^t \phi(s, X_s) dX_s$ by a martingale, not computing $\phi(s, X_s)$ too often

$X = W^{(d)}$ (d-dim. Brownian motion) or

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation
- Aim: Observe from the payoff function f “directly” the approximation properties of $f(X_1)$
- Simulate $\int_0^t \phi(s, X_s) dX_s$ by a martingale, not computing $\phi(s, X_s)$ too often

$X = W^{(d)}$ (d-dim. Brownian motion) or
$X = Y^{(d)}, Y_t^k = y_0^k + \sum_{m=1}^{d} \int_0^t \sigma_{km}(Y_u) dW_u^m$

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation
- Aim: Observe from the payoff function f “directly” the approximation properties of $f(X_1)$
- Simulate $\int_0^t \phi(s, X_s) dX_s$ by a martingale, not computing $\phi(s, X_s)$ too often

$X = W^{(d)}$ (d-dim. Brownian motion) or
$X = Y^{(d)}, Y_t^k = y_0^k + \sum_{m=1}^d \int_0^t \sigma_{km}(Y_u) dW_u^m$; $\sigma_{kk}(y) := y_k$

- Approximate stochastic integrals of the form

$$f(X_1) = \int_0^1 \phi(s, X_s) dX_s$$

  e.g. from option hedging
- Riemann approximation
- Aim: Observe from the payoff function f “directly” the approximation properties of $f(X_1)$
- Simulate $\int_0^t \phi(s, X_s) dX_s$ by a martingale, not computing $\phi(s, X_s)$ too often

$X = W^{(d)}$ (d-dim. Brownian motion) or
$X = Y^{(d)}$, $Y_t^k = y_0^k + \sum_{m=1}^{d} \int_0^t \sigma_{km}(Y_u) dW_u^m$; $\sigma_{kk}(y) := y_k$

Results for $d \geq 1$; consider $d = 1$ and $X = W$ for simplicity

- $(W_t)_{t\in[0,1]}$ standard Brownian motion, $(\mathcal{F}_t) = \overline{(\mathcal{F}_t^W)}$
- Let $f : \mathbb{R} \to \mathbb{R}$, $f(W_1) \in L_2$

- $(W_t)_{t\in[0,1]}$ standard Brownian motion, $(\mathcal{F}_t) = \overline{(\mathcal{F}_t^W)}$
- Let $f : \mathbb{R} \to \mathbb{R}$, $f(W_1) \in L_2$
- Define $F : [0,1] \times \mathbb{R} \to \mathbb{R} \in C^\infty([0,1[\times \mathbb{R})$,

$$F(t,x) := \mathbb{E}\left(f(W_1) \mid W_t = x\right) = \mathbb{E}f(x + W_{1-t}).$$

- $(W_t)_{t\in[0,1]}$ standard Brownian motion, $(\mathcal{F}_t) = \overline{(\mathcal{F}_t^W)}$
- Let $f : \mathbb{R} \to \mathbb{R}$, $f(W_1) \in L_2$
- Define $F : [0,1] \times \mathbb{R} \to \mathbb{R} \in C^\infty([0,1[\times \mathbb{R})$,

$$F(t,x) := \mathbb{E}(f(W_1) \mid W_t = x) = \mathbb{E}f(x + W_{1-t}).$$

Then

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} &= 0 \\ F(1,x) &= f(x). \end{aligned}$$

- $(W_t)_{t\in[0,1]}$ standard Brownian motion, $(\mathcal{F}_t) = \overline{(\mathcal{F}_t^W)}$
- Let $f : \mathbb{R} \to \mathbb{R}$, $f(W_1) \in L_2$
- Define $F : [0,1] \times \mathbb{R} \to \mathbb{R} \in C^\infty([0,1[\times \mathbb{R})$,

$$F(t,x) := \mathbb{E}\left(f(W_1) \mid W_t = x\right) = \mathbb{E}f(x + W_{1-t}).$$

Then

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} &= 0 \\ F(1,x) &= f(x). \end{aligned}$$

By Itô's formula:

$$f(W_1) = F(1, W_1) = \mathbb{E}f(W_1) + \int_0^1 \frac{\partial F}{\partial x}(s, W_s) dW_s.$$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}})\left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}})\left(W_{t_i \wedge t} - W_{t_{i-1}\wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}}) \left(W_{t_i \wedge t} - W_{t_{i-1}\wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}}) \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t} \right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s) dW_s - \sum_{i=1}^{n} \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}}) \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

- Properties of f $\longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$\|C_1(f,\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}}$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}}) \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$\|C_1(f,\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2}$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}})\left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$$\|C_1(f,\tau_n)\|_{L_2} \le \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2} \quad ; \mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{\sqrt{n}}\right) \le \tilde{c}\lambda^{-2}$$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}}) \left(W_{t_i \wedge t} - W_{t_{i-1}\wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$\|C_1(f,\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2}$; $\mathbb{P}\left(|C_1(f,\tau_n)| \geq \frac{\lambda}{\sqrt{n}}\right) \leq \tilde{c}\lambda^{-2}$

$\|C(f,\tau_n)\|_{BMO} \leq \frac{c}{\sqrt{n}} \iff f$ Lipschitz

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}})\left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$$\|C_1(f,\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2} \quad ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \geq \frac{\lambda}{\sqrt{n}}\right) \leq \tilde{c}\lambda^{-2}$$

$$\|C(f,\tau_n)\|_{BMO} \leq \frac{c}{\sqrt{n}} \iff f \text{ Lipschitz} \ ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \geq \frac{\lambda}{\sqrt{n}}\right) \leq \tilde{c}e^{-\lambda}$$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}})\left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$$\|C_1(f,\tau_n)\|_{L_2} \leq \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2} \quad ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \geq \frac{\lambda}{\sqrt{n}}\right) \leq \tilde{c}\lambda^{-2}$$

$$\|C(f,\tau_n)\|_{BMO} \leq \frac{c}{\sqrt{n}} \iff f \text{ Lipschitz} \ ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \geq \frac{\lambda}{\sqrt{n}}\right) \leq \tilde{c}e^{-\lambda}$$

$$\|C_1(f,\tau_n)\|_{L_2} \leq cn^{-\frac{\theta}{2}}$$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}}) \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right)$$

- Properties of f $\longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{B(R)}, \gamma)$

$$\|C_1(f,\tau_n)\|_{L_2} \le \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2} \quad ; \mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{\sqrt{n}}\right) \le \tilde{c}\lambda^{-2}$$

$$\|C(f,\tau_n)\|_{BMO} \le \frac{c}{\sqrt{n}} \iff f \text{ Lipschitz} ; \mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{\sqrt{n}}\right) \le \tilde{c}e^{-\lambda}$$

$$\|C_1(f,\tau_n)\|_{L_2} \le cn^{-\frac{\theta}{2}} \iff f \in B^{\theta}_{2,\infty}$$

Discretize the integral on $[0,1]$ using the time net $\tau_n := (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and get the approximation error:

$$C_t(f,\tau_n) := \int_0^t \frac{\partial F}{\partial x}(s, W_s)dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, W_{t_{i-1}})\left(W_{t_i \wedge t} - W_{t_{i-1}\wedge t}\right)$$

- Properties of $f \longleftrightarrow$ properties of $(C_t(f,\tau_n))_{t\in[0,1]}$
- How to measure the size of $(C_t(f,\tau_n))_{t\in[0,1]}$?
- Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$; $(\mathbb{R}, \overline{\mathcal{B}(\mathcal{R})}, \gamma)$

$$\|C_1(f,\tau_n)\|_{L_2} \le \frac{c}{\sqrt{n}} \iff f \in \mathbb{D}_{1,2} \quad ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{\sqrt{n}}\right) \le \tilde{c}\lambda^{-2}$$

$$\|C(f,\tau_n)\|_{BMO} \le \frac{c}{\sqrt{n}} \iff f \text{ Lipschitz} \ ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{\sqrt{n}}\right) \le \tilde{c}e^{-\lambda}$$

$$\|C_1(f,\tau_n)\|_{L_2} \le cn^{-\frac{\theta}{2}} \iff f \in B^{\theta}_{2,\infty} \quad ; \ \mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{n^{\frac{\theta}{2}}}\right) \le \tilde{c}\lambda^{-2}$$

Malliavin Sobolev space $\mathbb{D}_{1,2}$ and $\mathbb{D}_{1,p}$

$(\mathbb{R}, \overline{B(R)}, \gamma)$ probability space
$f : \mathbb{R} \to \mathbb{R}$, $f \in L_2$

Malliavin Sobolev space $\mathbb{D}_{1,2}$ and $\mathbb{D}_{1,p}$

$(\mathbb{R}, \overline{B(R)}, \gamma)$ probability space
$f : \mathbb{R} \to \mathbb{R}$, $f \in L_2$
$\mathbb{D}_{1,2}$ Malliavin Sobolev space
Df Malliavin derivative

Malliavin Sobolev space $\mathbb{D}_{1,2}$ and $\mathbb{D}_{1,p}$

$(\mathbb{R}, \overline{B(R)}, \gamma)$ probability space
$f : \mathbb{R} \to \mathbb{R}$, $f \in L_2$
$\mathbb{D}_{1,2}$ Malliavin Sobolev space
Df Malliavin derivative

$f \in \mathbb{D}_{1,p}$, $2 < p < \infty$

$$\|f\|_{\mathbb{D}_{1,p}} := \left(\|f\|_{L_p}^p + \|Df\|_{L_p}^p \right)^{\frac{1}{p}} < \infty$$

Malliavin Sobolev space $\mathbb{D}_{1,2}$ and $\mathbb{D}_{1,p}$

$(\mathbb{R}, \overline{B(R)}, \gamma)$ probability space
$f : \mathbb{R} \to \mathbb{R}$, $f \in L_2$
$\mathbb{D}_{1,2}$ Malliavin Sobolev space
Df Malliavin derivative

$f \in \mathbb{D}_{1,p}$, $2 < p < \infty$
$\|f\|_{\mathbb{D}_{1,p}} := \left(\|f\|_{L_p}^p + \|Df\|_{L_p}^p \right)^{\frac{1}{p}} < \infty$
Examples:

$$f_1(x) = \max(0, x), \quad f_1 \in \mathbb{D}_{1,p} \text{ for all } p \geq 2$$

Malliavin Sobolev space $\mathbb{D}_{1,2}$ and $\mathbb{D}_{1,p}$

$(\mathbb{R}, \overline{B(R)}, \gamma)$ probability space
$f : \mathbb{R} \to \mathbb{R}$, $f \in L_2$
$\mathbb{D}_{1,2}$ Malliavin Sobolev space
Df Malliavin derivative

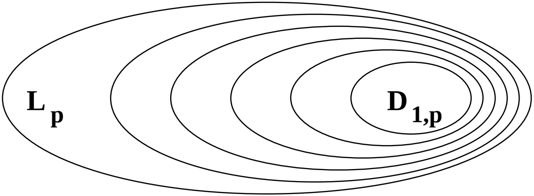
$f \in \mathbb{D}_{1,p}$, $2 < p < \infty$
$\|f\|_{\mathbb{D}_{1,p}} := \left(\|f\|_{L_p}^p + \|Df\|_{L_p}^p \right)^{\frac{1}{p}} < \infty$
Examples:

$$f_1(x) = \max(0, x), \quad f_1 \in \mathbb{D}_{1,p} \text{ for all } p \geq 2$$

$$f_\alpha(x) = \max(0, x^\alpha), \quad \frac{1}{2} < \alpha < 1 : f_\alpha \in \mathbb{D}_{1,p} \text{ for } p < \frac{1}{1-\alpha}$$

Fractional smoothness of order $0 < \theta < 1$ for $2 \leq p < \infty$:

Besov space $B^{\theta}_{p,q} = (L_p, \mathbb{D}_{1,p})_{\theta,q}$ $\qquad (1 \leq q \leq \infty)$

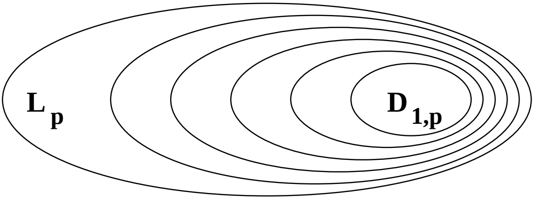


$\theta = 0 \sim L_p$, $\theta = 1 \sim \mathbb{D}_{1,p}$

lexicographical order in θ, q

Fractional smoothness of order $0 < \theta < 1$ for $2 \leq p < \infty$:

Besov space $B^{\theta}_{p,q} = (L_p, \mathbb{D}_{1,p})_{\theta,q}$ $\qquad (1 \leq q \leq \infty)$



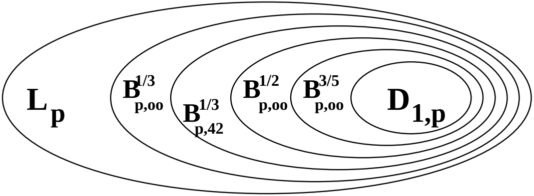
$\theta = 0 \sim L_p$, $\theta = 1 \sim \mathbb{D}_{1,p}$ lexicographical order in θ, q

$$\|g\|_{B^{\theta}_{p,q}} := \left(\int_0^{\infty} \left| t^{-\theta} \inf \left\{ \|g_0\|_{L_p} + t\|g_1\|_{\mathbb{D}_{1,p}} : g = g_0 + g_1 \right\} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$\|g\|_{B^{\theta}_{p,\infty}} := \sup_{t>0} \left| t^{-\theta} \inf \left\{ \|g_0\|_{L_p} + t\|g_1\|_{\mathbb{D}_{1,p}} : g = g_0 + g_1 \right\} \right|$$

Fractional smoothness of order $0 < \theta < 1$ for $2 \leq p < \infty$:

$$\text{Besov space } B^{\theta}_{p,q} = (L_p, \mathbb{D}_{1,p})_{\theta,q} \qquad (1 \leq q \leq \infty)$$



$\theta = 0 \sim L_p$, $\theta = 1 \sim \mathbb{D}_{1,p}$ lexicographical order in θ, q

$$\|g\|_{B^{\theta}_{p,q}} := \left(\int_0^{\infty} \left| t^{-\theta} \inf \left\{ \|g_0\|_{L_p} + t\|g_1\|_{\mathbb{D}_{1,p}} : g = g_0 + g_1 \right\} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$\|g\|_{B^{\theta}_{p,\infty}} := \sup_{t>0} \left| t^{-\theta} \inf \left\{ \|g_0\|_{L_p} + t\|g_1\|_{\mathbb{D}_{1,p}} : g = g_0 + g_1 \right\} \right|$$

Examples:

$$f_0(x) = \chi_{[0,\infty[}(x)\,,\, f_0 \in B_{p,\infty}^{\frac{1}{p}} \text{ for all } p \geq 2$$

Examples:

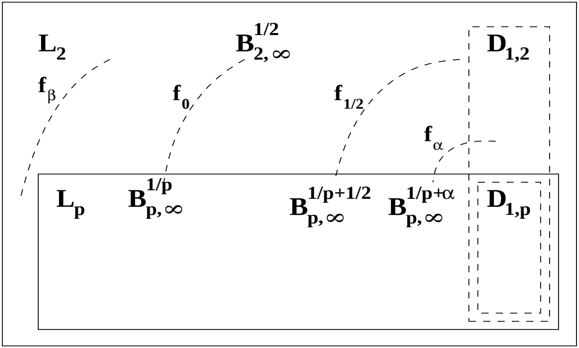
$$f_0(x) = \chi_{[0,\infty[}(x)\,,\, f_0 \in B^{\frac{1}{p}}_{p,\infty} \text{ for all } p \geq 2$$

$$f_\alpha(x) = \max(0, x^\alpha), 0 < \alpha < 1 : f_\alpha \in B^{\frac{1}{p}+\alpha}_{p,\infty} \text{ for } p > \frac{1}{1-\alpha}$$

Examples:

$$f_0(x) = \chi_{[0,\infty[}(x)\,,\, f_0 \in B^{\frac{1}{p}}_{p,\infty} \text{ for all } p \geq 2$$

$$f_\alpha(x) = \max(0, x^\alpha), 0 < \alpha < 1 : f_\alpha \in B^{\frac{1}{p}+\alpha}_{p,\infty} \text{ for } p > \frac{1}{1-\alpha}$$



Theorem 1 (Equidistant time net)

$2 \leq p < \infty$, $0 < \theta < 1$

$$f \in B^{\theta}_{p,\infty} \iff \left|\left|C_1(f, \tau^1_n)\right|\right|_p \leq cn^{-\frac{\theta}{2}}$$

- $p = 2$: Geiss-Geiss, 2004 and Geiss-Hujo, 2007

Theorem 2

$2 \leq p < \infty$, $0 < \theta < 1$, $1 \leq q \leq \infty$

$$\begin{aligned}
\|f\|_{B^{\theta}_{p,q}} &\sim_c \|f\|_p + \left\| (1-t)^{-\frac{\theta}{2}} \|F(1,W_1) - F(t,W_t)\|_p \right\|_{L_q([0,1),\frac{dt}{1-t})} \\
&\sim_c \|f\|_p + \left\| (1-t)^{\frac{1-\theta}{2}} \left\| \frac{\partial F}{\partial x}(t,W_t) \right\|_p \right\|_{L_q([0,1),\frac{dt}{1-t})} \\
&\sim_c \|f\|_p + \left\| (1-t)^{\frac{2-\theta}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t,W_t) \right\|_p \right\|_{L_q([0,1),\frac{dt}{1-t})}
\end{aligned}$$

Theorem 2

$2 \leq p < \infty$, $0 < \theta < 1$, $1 \leq q \leq \infty$

$$
\begin{aligned}
\|f\|_{B^{\theta}_{p,q}} &\sim_c \|f\|_p + \left\| (1-t)^{-\frac{\theta}{2}} \|F(1,W_1) - F(t,W_t)\|_p \right\|_{L_q([0,1),\frac{dt}{1-t})} \\
&\sim_c \|f\|_p + \left\| (1-t)^{\frac{1-\theta}{2}} \left\| \frac{\partial F}{\partial x}(t,W_t) \right\|_p \right\|_{L_q([0,1),\frac{dt}{1-t})} \\
&\sim_c \|f\|_p + \left\| (1-t)^{\frac{2-\theta}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t,W_t) \right\|_p \right\|_{L_q([0,1),\frac{dt}{1-t})}
\end{aligned}
$$

Theorem 3

$2 \leq p < \infty$, $0 < \theta < 1$

$$
\|C_1(f,\tau_n)\|_p \sim_c \left\| \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - u) \left[\frac{\partial^2 F}{\partial x^2}(u,W_u) \right]^2 du \right)^{\frac{1}{2}} \right\|_p
$$

Adapted time nets, $0 < \theta < 1$

Define

$$|\tau|_\theta := \sup_{i=1,\ldots,n} \sup_{t_{i-1} \le u < t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}}$$

Adapted time nets, $0 < \theta < 1$

Define

$$|\tau|_\theta := \sup_{i=1,\ldots,n} \sup_{t_{i-1} \le u < t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}}$$

- $|\tau|_1 = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$

Adapted time nets, $0 < \theta < 1$

Define

$$|\tau|_\theta := \sup_{i=1,\dots,n} \sup_{t_{i-1} \le u < t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}}$$

- $|\tau|_1 = \max_{i=1,\dots,n} |t_i - t_{i-1}|$

Define

$$\tau_n^\theta := \left(t_i^{n,\theta}\right)_{i=0}^{n} \text{ with } t_i^{n,\theta} := 1 - (1 - \frac{i}{n})^{\frac{1}{\theta}}$$

Adapted time nets, $0 < \theta < 1$

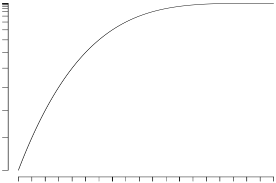
Define

$$|\tau|_\theta := \sup_{i=1,\dots,n} \sup_{t_{i-1} \le u < t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}}$$

- $|\tau|_1 = \max_{i=1,\dots,n} |t_i - t_{i-1}|$

Define

$$\tau_n^\theta := \left(t_i^{n,\theta} \right)_{i=0}^{n} \text{ with } t_i^{n,\theta} := 1 - (1 - \frac{i}{n})^{\frac{1}{\theta}}$$



Adapted time nets, $0 < \theta < 1$

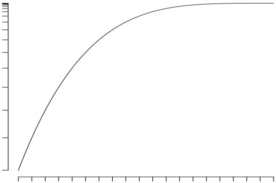
Define

$$|\tau|_\theta := \sup_{i=1,\dots,n} \sup_{t_{i-1} \le u < t_i} \frac{|t_i - u|}{(1-u)^{1-\theta}}$$

- $|\tau|_1 = \max_{i=1,\dots,n} |t_i - t_{i-1}|$

Define

$$\tau_n^\theta := \left(t_i^{n,\theta}\right)_{i=0}^{n} \text{ with } t_i^{n,\theta} := 1 - (1 - \frac{i}{n})^{\frac{1}{\theta}}$$



- $\left|\tau_n^\theta\right|_\theta \le \frac{1}{\theta n}$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$f \in D_p^\theta \iff \left\| C_1(f, \tau_n^\theta) \right\|_p \leq c n^{-\frac{1}{2}}$$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$f \in D_p^\theta \iff \left\| C_1(f, \tau_n^\theta) \right\|_p \leq c n^{-\frac{1}{2}}$$

$$\iff \| C_1(f, \tau) \|_p \leq c \left\| |\tau|^{\frac{1}{2}}_{\theta} \right\|_\infty$$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$f \in D_p^\theta \iff \left\| C_1(f, \tau_n^\theta) \right\|_p \leq c n^{-\frac{1}{2}}$$

$$\iff \| C_1(f, \tau) \|_p \leq c \left\| |\tau|^{\frac{1}{2}}_{\theta} \right\|_\infty$$

$$f \in D_p^\theta \ : \ \left\| \left(\int_0^1 (1-u)^{1-\theta} \left[\frac{\partial^2 F}{\partial x^2}(u, W_u) \right]^2 du \right)^{\frac{1}{2}} \right\|_p < \infty$$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$f \in D_p^\theta \iff \left\| C_1(f, \tau_n^\theta) \right\|_p \leq c n^{-\frac{1}{2}}$$

$$\iff \| C_1(f, \tau) \|_p \leq c \left\| |\tau|^{\frac{\frac{1}{2}}{\theta}} \right\|_\infty$$

$$f \in D_p^\theta \ : \ \left\| \left(\int_0^1 (1-u)^{1-\theta} \left[\frac{\partial^2 F}{\partial x^2}(u, W_u) \right]^2 du \right)^{\frac{1}{2}} \right\|_p < \infty$$

- $D_2^\theta = B_{2,2}^\theta$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$
\begin{aligned}
f \in D_p^\theta \quad &\Longleftrightarrow \quad \left\| C_1(f, \tau_n^\theta) \right\|_p \leq c n^{-\frac{1}{2}} \\
&\Longleftrightarrow \quad \| C_1(f, \tau) \|_p \leq c \left\| |\tau|^{\frac{\frac{1}{2}}{\theta}} \right\|_\infty
\end{aligned}
$$

$$
f \in D_p^\theta \ : \ \left\| \left(\int_0^1 (1-u)^{1-\theta} \left[\frac{\partial^2 F}{\partial x^2}(u, W_u) \right]^2 du \right)^{\frac{1}{2}} \right\|_p < \infty
$$

- $D_2^\theta = B_{2,2}^\theta$, $\quad B_{p,2}^\theta \subset D_p^\theta \subset B_{p,\infty}^\theta$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$f \in D_p^\theta \iff \left|\left|C_1(f, \tau_n^\theta)\right|\right|_p \leq cn^{-\frac{1}{2}}$$
$$\iff \|C_1(f, \tau)\|_p \leq c \left|\left||\tau|^{\frac{1}{2}}_{\theta}\right|\right|_\infty$$

$$f \in D_p^\theta \ : \ \left|\left|\left(\int_0^1 (1-u)^{1-\theta} \left[\frac{\partial^2 F}{\partial x^2}(u, W_u)\right]^2 du\right)^{\frac{1}{2}}\right|\right|_p < \infty$$

- $D_2^\theta = B_{2,2}^\theta$, $B_{p,2}^\theta \subset D_p^\theta \subset B_{p,\infty}^\theta$, $D_p^1 = \mathbb{D}_{1,p}$

Theorem 4 (Non-equidistant time nets)

$2 \leq p < \infty$, $0 < \theta \leq 1$

$$f \in D_p^\theta \iff \left\| C_1(f, \tau_n^\theta) \right\|_p \leq c n^{-\frac{1}{2}}$$

$$\iff \|C_1(f,\tau)\|_p \leq c \left\| |\tau|^{\frac{1}{2}}_{\theta} \right\|_\infty$$

$$f \in D_p^\theta \; : \; \left\| \left(\int_0^1 (1-u)^{1-\theta} \left[\frac{\partial^2 F}{\partial x^2}(u, W_u) \right]^2 du \right)^{\frac{1}{2}} \right\|_p < \infty$$

- $D_2^\theta = B_{2,2}^\theta$, $B_{p,2}^\theta \subset D_p^\theta \subset B_{p,\infty}^\theta$, $D_p^1 = \mathbb{D}_{1,p}$
- $p = 2$: Geiss-Hujo, 2007

- S. Geiss. Quantitative approximation of certain stochastic integrals. *Stochastics and Stochastics Reports*, 73:241-270, 2002.
- C. Geiss and S. Geiss. On approximation of a class of stochastic integrals and interpolation. *Stochastics and Stochastics Reports*, 76:339-262, 2004.
- S. Geiss. Weighted BMO and discrete time hedging within the Black-Scholes model. *Prob. Theory Rel. Fields*, 2004.
- S. Geiss and M. Hujo. Interpolation and approximation in $L_2(\gamma)$. *Journal of Approximation Theory*, 2007.
- E. Gobet and E. Temam. Discrete time hedging errors for options with irregular payoffs. *Finance and Stochastics*, 2001.
- A. Toivola. Interpolation and approximation in L_p. Preprint 380, Dept. of Mathematics and Statistics, University of Jyväskylä, 2009.
- A. Toivola. On fractional smoothness and approximations of stochastic integrals. Report 123, Rep. Univ. Jyväskylä Dept. Math. Stat, 2009.