SCALLABILITY ANALYSIS OF PARALLEL MIC(0) PRECONDITIONING ALGORITHM FOR 3D ELLIPTIC PROBLEMS.

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ABSTRACT. Novel parallel algorithms for the solution of large FEM linear systems arising from second order elliptic partial differential equations in 3D are presented. The problem is discretized by rotated trilinear nonconforming Rannacher–Turek finite elements. The resulting symmetric positive definite system of equations $A\mathbf{x} = \mathbf{f}$ is solved by the preconditioned conjugate gradient algorithm. The preconditioners employed are obtained by the modified incomplete Cholesky factorization MIC(0) of two kinds of auxiliary matrices B that both are constructed as local approximations of A. Two parallel algorithms based on the different block structures of the related matrices B are studied. The numerical tests presented confirm the scallability of one of the algorithms.

1. INTRODUCTION

We consider the model elliptic boundary value problem:

$$
\begin{aligned}
Lu \equiv -\nabla \cdot (a(x)\nabla u(x)) &= f(x) \text{ in } \Omega, \\
u &= 0 \text{ on } \Gamma_D, \\
(a(x)\nabla u(x)) \cdot n &= 0 \text{ on } \Gamma_N,
\end{aligned}
\tag{1.1}
$$

where $\Omega = [0,1]^3 \subset \mathbb{R}^3$, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $a(x)$ is a symmetric and positive definite coefficient matrix. The problem is discretized using non-conforming finite elements method (FEM). The resulting linear algebraic system is assumed to be large. The stiffness matrix A is symmetric and positive definite. For large scale problems, the preconditioned conjugate gradient (PCG) method is known to be the best solution method [**?**].

The recent efforts in development of efficient solution methods for non-conforming finite element systems is inspired by their importance for various applications in scientific computations and engineering [14, 2, 13]. The goal of this study is to develop new parallel PCG solvers for the arising 3D FEM elliptic systems. Locally modified approximations of the global stiffness matrix are proposed allowing for: a) a stable MIC(0) (modified incomplete Cholesky) factorization; and b) a scalable parallel implementation. The considered non-conforming FEM and MIC(0) factorization are robust for problems with possible jumps of the coefficients

The algorithm is based on the experience in developing such kind of algorithms for 2D problems using conforming FEM elements on skewed meshes [10] and non-conforming rotated bilinear FEM elements [**?**, 13, **?**]. The rotated trilinear non-conforming finite elements on hexahedrons are used for the numerical solution of (1.1).

We assume that $\Omega^h = w_1^{h_1} \times w_2^{h_2} \times w_3^{h_3}$ is a decomposition of the computational domain $\Omega \subset \mathbb{R}^3$ into hexahedrons. The degrees of freedom are associated with the midpoints of the sides. The standard computational procedure leads to the linear system of equations $Ax = b$, where the stiffness matrix A is sparse, symmetric and positive definite.

The rest of this paper is organised as follows. Section 2 describes the element-by-element construction of the preconditioners. Section 3 contains the parallel implementation details and estimates of parallel times. Some results from numerical experiments, are presented in Section 4.

Table 1. Variant B1

n	N	p	I	T_{C1}	T_{C2}	T_{C3}
127	6193536	1	44	96.21	133.9	85.64
160	12364800	2	50	128.5	186.7	94.74
202	24849636	4	56	197.6	258.7	127.92
255	49939200	8	64	317.7	361.2	182.57
322	100469796	16	72	465.4	608.8	289.59
406	201264756	32	81	759.9	1072.	435.69

Table 2. Variant B2

n	N	p	I	T_{C1}	T_{C2}	T_{C3}
127	6193536	1	44	68.88	125.15	94.17
160	12364800	2	49	79.91	189.36	109.82
202	24849636	4	54	93.74	225.12	127.63
255	49939200	8	61	114.4	287.38	153.52
322	100469796	16	68	163.4	368.83	252.48
406	201264756	32	76	165.5	480.11	245.04

Programs for Variants B1 and B2 access the memory in a different pattern. This explains the different behavior of sequential times, comparing variants B1 and B2 on different machines. The tables well illustrate the different properties of the computing platforms used.

As expected, one can observe that the iteration count is of order $O(n^{1/2}) = O(N^{1/6})$ and the total time grows as $O(n^{1/2}) = O(N^{1/6})$, especially in Variant B2.

Acknowledgments. The partial support through EC INCO Grant BIS-21++ 016639/2005 is gratefully acknowledged. The numerical tests are supported via EC Project HPC-EUROPA RII3-CT-2003-506079.

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