Master's Thesis
Stability of nonlinear subdivision schemes and multiresolutions

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1 Introduction

Subdivision is a process of recursively refining discrete data using a set of subdivision rules (called subdivision scheme) which generates a continuous or even smooth limit. It has numerous applications, such as image reconstruction, the design of curves and surfaces, shape preservation in data and geometric objects, the approximation of arbitrary functions, etc. On the other hand, subdivision lies in the core of multiresolution analysis (MRA) and wavelet transforms, and thus plays a central role in data compression, noise removal, and so on. The great variety of applications as well as the necessity of improving the performance of the existent algorithms lead to the invention of a great variety of subdivision schemes. The main mathematical issues to be investigated for a subdivision scheme are convergence as the number of subdivision steps goes to infinity, the smoothness of the limit objects, and the stability of the subdivision process and the associated multiresolution representations. In the linear, stationary case, due to standard techniques [CDM91], [Dyn92], there is a complete theory about convergence and stability of both subdivision and multiresolution, as well as about determining the smoothness of the first. In the nonlinear case, however, there are still very few results about stability and smoothness. This is due to the fact that the theorems from the linear setup can not be directly extended to the nonlinear one. For example, there are weakly nonlinear subdivision schemes such as the essentially non-oscillatory (ENO) scheme [CDM03] which are uniformly convergent but not stable. Moreover, in linear subdivision, stability of the scheme guarantees stability of the corresponding multiresolution. However, in this paper we will show that certain convergent schemes for which subdivision is stable (e.g., the dyadic median-interpolating scheme) do not possess a stable associated pyramid transform. In summary, analyzing subdivision and the associated multiresolution is a very active research field with many open problems, that is driven by both theory and applications.

Let us briefly introduce some notions and fix the notation. For details see Section 2. In every subdivision scheme, a family of "nested" grids $\Gamma^j, j \geq 0$ is given (i.e., $\Gamma^i \subset \Gamma^{i+1}, \forall i$), the initial data v^0 is a uniformly bounded sequence on the coarsest grid ($v^0 \in l_\infty(\Gamma^0)$), and the subdivision operators $S^{[j]} : l_\infty(\Gamma^j) \to l_\infty(\Gamma^{j+1})$ map sequences corresponding to two consecutive grids into each other. The grids may be finite or infinite, and data can be arbitrary. We will denote by v^j the data sequence on Γ^j, generated by v^0 and j subdivision steps. The subdivision process is *local*, respectively *regular*, respectively *univariate* if the operator $S^{[j]}$ is built from local rules, respectively the grids $\{\Gamma^i\}_{i=0}^\infty$ are uniform, respectively univariate. It is *stationary*, if $S^{[j]} \equiv S, \forall j \in \mathbb{N}$, and *linear* if all operators $S^{[j]}$ are linear. We are interested only in stationary subdivision, and all the schemes we consider are of that type. Moreover, if the subdivision operator is in addition shift-invariant the analysis can be performed on $l_\infty(\mathbb{Z})$. The locality of S assures that changing the initial data at one place will have a local effect on the limit function, i.e., the new limit will differ from the original one only in a neighborhood of the initial point. In that case we can further restrict our analysis of the scheme to a finite-dimensional subspace of $l_\infty(\mathbb{Z})$.

Multiresolution, as defined by Harten in [Har96], is a pyramid transform that, starting from fine-scale data, constructs data on coarser scales by restriction and so called "details" (analysis step) and conversely reconstructs the original data from their coarse-scale part and the details (synthesis step). More precisely, for a given $J \in \mathbb{N}$ we define a set of grids $\{\Gamma^j\}_0^J$, a set of restriction operators $R^{[j]} : l_\infty(\Gamma^{j+1}) \to l_\infty(\Gamma^j), j = 0, \dots, J-1$, as well as subdivision operators $\{S^{[j]}\}_0^{J-1}$ such that $R^{[j]}S^{[j]} = I_j, \forall j = 0, \dots, J-1$ where I_j is the identity operator on $l_\infty(\Gamma^j)$. Thus, one can uniquely decompose every sequence $v^J \in l_\infty(\Gamma^J)$ into its coarse-scale part $v^0 = R^{[0]}R^{[1]}\cdots R^{[J-1]}v^J$ and a sequence of *details* $d^j \in l_\infty(\Gamma^j)$, $j = 1, \ldots, J$, such that there is a correspondence

$$v^J \qquad \longleftrightarrow \qquad \{v^0, d^1, \ldots, d^J\}.$$

Again, we are interested only in stationary shift-invariant multiresolution analysis, i.e., the subdivision and the restriction operators are not level-dependent and can be modeled as operators on $l_\infty(\mathbb{Z})$, and we denote them by S and R respectively. In most of the applications the detail sequences $\{d^j\}$ are linked with the sequence of coarse-scale representations $\{v^j\}$ via the formulas $v^j = Sv^{j-1} + d^j$ and $v^j = R^{J-j}v^J$, $j \leq J$. There are more general ways of defining these sequences, see for instance [DRS04] or [URDS$^+$05], but we will not go into these generalizations for the moment being. We will concentrate at the synthesis step of the multiresolution analysis, and will refer to it as *the multiresolution scheme associated to S*. We introduce the coarse-to-fine grid operator M by the formula $v^j = Mv^{j-1} := Sv^{j-1} + d^j$. Note that, although M depends on two sequences v^{j-1} and d^j, we have chosen not to indicate the argument d^j to keep the exposition as easy to follow as possible. M is *linear* if S is linear.

We define Lipschitz stability of a subdivision scheme, given by S, and its corresponding multiresolution scheme, given by M as follows: There exists a constant C such that

$$\|S^J v^0 - S^J \tilde{v}^0\| \leq C\|v^0 - \tilde{v}^0\|, \tag{1.1}$$

$$\|M^J v^0 - M^J \tilde{v}^0\| \leq C\Big(\|v^0 - \tilde{v}^0\| + \sum_{j=1}^{J} \|d^j - \tilde{d}^j\|\Big), \tag{1.2}$$

for all $J \in \mathbb{N}$ and all sequences involved.

The following example helps to better understand the above definitions. Consider the subdivision scheme S_c based on central Lagrange cubic interpolation. More precisely, let $\Gamma^j = 2^{-j}\mathbb{Z}$ and define $S_c v^j$ as follows: The values at the "old" grid points $2^{-j}k$ belonging to $\Gamma^{j+1} \cap \Gamma^j$ are inherited from the corresponding entries of v^j (i.e., S_c is *interpolatory*) while the value at "new" grid points $2^{-(j+1)}(2k+1)$ is given by $p_k(2^{-(j+1)}(2k+1))$, where p_k is a cubic polynomial that interpolates v^j at the four points $2^{-j}l, l = k-1, \ldots, k+2$ neighboring it (fig. 1a). Since $2^{-1}\mathbb{Z} \cong \mathbb{Z}$ and using dilation, we have defined an operator $S_c : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$. Direct computations lead to the following explicit formula

$$\begin{aligned} v^{j+1}_{2k} &= (S_c v^j)_{2k} = v^j_k \\ v^{j+1}_{2k+1} &= (S_c v^j)_{2k+1} = -\frac{1}{16}v^j_{k-1} + \frac{9}{16}v^j_k + \frac{9}{16}v^j_{k+1} - \frac{1}{16}v^j_{k+2}, \end{aligned} \tag{1.3}$$

where v^j_k is the value of v^j at $2^{-j}k$ for every j, k. S_c is linear and local, since the entries of v^{j+1} are finite linear combinations of those of v^j; univariate and regular, because the grids $\{\Gamma^j\}$ are equidistant subsets of $\mathbb{R}$; stationary and shift-invariant, as the coefficients in (1.3) do not depend on the subdivision level j and the position k; and dyadic, because of dilation factor 2. Locality and shift-invariance enable us to represent the subdivision operator S_c by a single finitely supported sequence a, called *mask*, such that

$$(S_c v)_k = \sum_l a_{k-2l} v_l, \qquad k \in \mathbb{Z}. \tag{1.4}$$

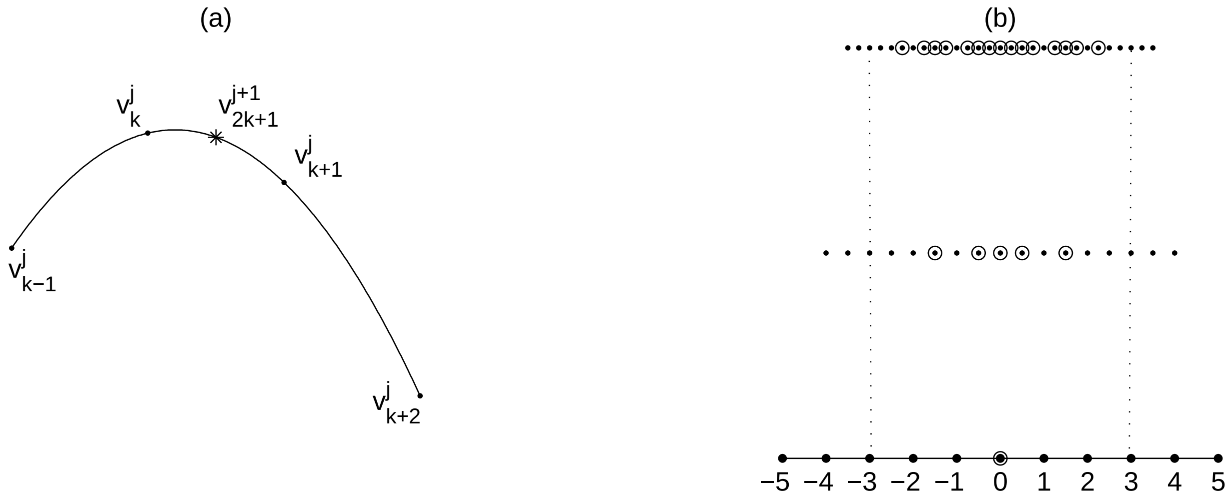

Figure 1: (a) The action of S_c, (b) The range of the dependence of v_0^0 after two subdivision steps.

The only nonzero entries of a are $a_{-3} = a_3 = -\frac{1}{16}$; $a_{-1} = a_1 = \frac{9}{16}$; $a_0 = 1$. Fig. 1b shows that changing the data at v_0^0 affects the limit function only in the interval $[-3, 3]$. On the other hand, analyzing the behavior of the limit in $[0, 1]$ characterizes it globally on $\mathbb{R}$ and thus, instead of $l_\infty(\mathbb{Z})$ one can work with sequences on $\{-2, -1, 0, 1, 2, 3\}$.

There are many cases in which linear multiresolution gives unsatisfactory results, and nonlinear alternatives are necessary. Tentatively, we can group the so-far proposed nonlinear subdivision schemes and MRAs into four: schemes that are shape preserving or capture singularities, normal multiresolution, statistical and morphological pyramids, and manifold subdivision. The first group contains of schemes that deal with Gibbs-type phenomena that are typical for linear schemes near jump singularities in the data [HEOC87], [LOC94], [SM04], or address shape preservation (monotonicity, convexity, etc.) without loss of smoothness in the limit [KvD99], [KvD97]. Normal meshes and multiresolution [GVSS00] deal with the efficient encoding of curves and surfaces, and nonlinearity in these schemes appears in the way details are computed, because they are of different type than the initial data. Examples from the third group have been proposed in connection with removing heavily-tailed (Cauchy) noise, nonlinearity here results mainly from the use of nonlinear robust estimators, such as the median [DY00]. The nonlinearity in the last group comes from the restriction of the control points to a manifold, surface, or a Lie group, this is a nonlinearly constrained set in the ambient space [URDS$^+$05], [WD05]. Although most of the above references provide analysis only in the univariate case, most of the schemes have natural multivariate analogues and have been introduced with view towards multivariate applications.

Our master thesis develops a general theory for stability of univariate nonlinear subdivision and multiresolution schemes. The paper is organized as follows: Section 2 sets the notation and contains all the definitions that we will use later on. Section 3 reviews the existing results, and provides references to the work of other authors in the field of nonlinear subdivision and multiresolution. Section 4 covers the theory in the linear setting and closely follows [Dyn92]. In Section 5 we introduce in details the nonlinear univariate subdivision schemes, which we will analyze later. Section 6 is our main contribution. It contains our stability theorem, its joint spectral radius version, and analyzes the conditions in the theorem. The applications of this theory are in Section 7. Section 8 is for conclusions.

2 Notation and basic definitions

2.1 Subdivision

In this paper we deal only with stationary subdivision schemes, and thus, we will not write the word "stationary" anymore. Note that in the introduction we discussed that in order the subdivision scheme S to be stationary, the subdivision process should be regular. Therefore we consider only regular subdivision as well. Moreover, we always use the space of bounded sequences l_∞ in the definition of subdivision.

Let us give some basic definitions.

Definition 2.1 *A subdivision scheme S is **finite** if there exists an integer $B \geq 0$, such that for every $\alpha \in \mathbb{Z}^s$, $(v^1)_\alpha$ is a function of at most B elements of v^0.*

Definition 2.2 *A subdivision scheme S **reproduces constants** if $S\mathbf{1} = \mathbf{1}$, where $\mathbf{1} \in l_\infty(\mathbb{Z}^s)$ is the constant sequence 1.*

Definition 2.3 *A subdivision scheme S is **affine invariant** if $S(av + b\mathbf{1}) = aSv + b\mathbf{1}$ for every $a, b \in \mathbb{R}$ and $v \in l_\infty(\mathbb{Z}^s)$.*

In most of the papers concerning subdivision, the entries of the sequence v^0 are interpreted as data over the grid $\mathbb{Z}^s$. Intuitively, we think of $(v^0)_\alpha$ as a point in $\mathbb{R}^{s+1}$ with coordinates $(\alpha, (v^0)_\alpha)$, where $\alpha \in \mathbb{Z}^s$ is a multi-index.

Definition 2.4 *Let $S : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ be a subdivision rule. The associate subdivision scheme S is **shift-invariant** if there exists an $s \times s$ integer matrix D with $\lim_{n\to\infty} D^{-n} = 0$, such that*

$$S \circ T_\alpha = T_{D\alpha} \circ S, \qquad \alpha \in \mathbb{Z}^s, \tag{2.1}$$

where $(T_\alpha v)_\beta = v_{\alpha+\beta}$ is the translation operator.
*D is called the **dilation matrix**, or simply **dilation factor**, when $s = 1$, of S.*

We will work only with diagonal dilation matrices $D = diag\{r_1, r_2, \ldots, r_s\}$, where $r_i \geq 2,\ i = 1, \ldots, s$, and we will say that "$S$ has dilation factor $r = (r_1, \ldots, r_s) \in \mathbb{Z}^s$" instead of "$S$ has dilation matrix D". Note that, due to $T_{\alpha+\beta} = T_\alpha \circ T_\beta$, it suffices to check (2.1) only for a basis of $\mathbb{Z}^s$. Subdivision schemes with dilation factor 2 are called *dyadic*, and those with dilation factor 3 - *triadic*.

Due to avoiding technicalities, we will set $s = 1$ in the following definition. Its generalization in higher dimensions is obvious and is left to the reader.

Definition 2.5 *Let $S : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ be a subdivision rule. The associate subdivision scheme S is **local**, if it is finite, shift-invariant with dilation factor r, and for every $i \in \mathbb{Z}$, $(v^1)_i$ depends only on $(v^0)_{[i/r]+j}$, $j \in J$, where $J = \{[a,b] : a, b \in \mathbb{Z}, b - a \leq 2B - 1\}$ contains 0.*

In this paper we consider only local subdivision schemes on $\mathbb{Z}$. All the norms that we use are *infinity-norms* and hence we will simply denote them by $\|\cdot\|$. Whether the norm is on an operator, on a sequence, or on a function will be clear from the context. We denote by $p^j \in l_\infty(\mathbb{Z})$ the sequence that corresponds to the values of the polynomial $p(x)$ on the grid Γ^j, for example if $\Gamma^j = r^{-j}\mathbb{Z}$, then $p^j_k = p(r^{-j}k)\ \ \forall k \in \mathbb{Z}$. We denote by $\Pi_n(\mathbb{R})$ the space of all polynomials with real coefficients of degree less or equal to n. We denote, by $\triangle^n$ the *n-th order forward finite difference operator*, i.e. for $v \in l_\infty(\mathbb{Z})$

$$(\triangle^n v)_k = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} v_{k+m}.$$

Now we are ready to define the basic notions of convergence and stability of a subdivision scheme. We will use the definitions in [Dyn92], and [CDM03].

Definition 2.6 *Let S be regular, shift-invariant subdivision scheme, with dilation factor $r \in \mathbb{Z}^s$. For every $v^0 \in l_\infty(\mathbb{Z}^s)$ and every $j \in \mathbb{Z}_+ \cup \{0\}$ define $f^j \in C(\mathbb{R}^s)$ as the piecewise linear function that interpolates the data $v^j = S^j v^0$ over Γ^j, and $g^j \in L_\infty(\mathbb{R}^s)$ as the piecewise constant function with $g^j(x) = (v^j)_\alpha, x \in \prod_{i=1}^s r_i^{-j}[\alpha_i, \alpha_i + 1]$.*

Definition 2.7 *Let $S : l_\infty(\mathbb{Z}^s) \to l_\infty(\mathbb{Z}^s)$ be a regular, shift-invariant subdivision rule with dilation factor $r \in \mathbb{Z}^s$.*

*(i) S is **convergent** if for every sequence $v^0 \in l_\infty(\mathbb{Z}^s)$, there exists a continuous function $f \in C(\mathbb{R}^s)$ such that*

$$\lim_{j<J\to\infty} |(S^J v^0)_{r^{J-j}\alpha} - f(r^{-j}\alpha)| = 0, \quad \alpha \in \mathbb{Z}^s,\ \ j \in \mathbb{Z}_+,$$

and such that the operator

$$S^\infty : l_\infty(\mathbb{Z}^s) \to C(\mathbb{R}^s), \qquad S^\infty v^0 = f,$$

is not trivially zero.

*(ii) S is **uniformly convergent** if for every sequence $v^0 \in l_\infty(\mathbb{Z}^s)$*

$$\lim_{J\to\infty} \|S^J v^0 - f(\frac{\cdot}{r^J})\| = 0, \tag{2.2}$$

where $f(r^{-J}\cdot)$ denotes the sequence $\{f(\alpha/r^J) : \alpha \in \mathbb{Z}^s\}$, and S^∞ is a non-trivial map.

Here, by $\alpha\beta$ for $\alpha, \beta \in \mathbb{Z}^s$ we understand $\gamma \in \mathbb{Z}^s$, such that $\gamma_i = \alpha_i\beta_i,\ \ i = 1 \dots s$.

Thus, a subdivision scheme is convergent iff $\{g^j\}_{j=0}^{\infty}$ converges pointwise in $L_\infty(\mathbb{R})$ to a continuous function f, and uniformly convergent iff both $\{f^j\}_{j=0}^{\infty}$ and $\{g^j\}_{j=0}^{\infty}$ converge uniformly in $L_\infty(\mathbb{R})$ to f (due to uniform convergence, the limit is automatically continuous). There are, of course, other $\{h^j(x)\}_{j=0}^{\infty}$ based on v^k that converge uniformly to f, but we will not use them in this paper. A complete classification of such functional sequences is done in Lemma 2.3. in [Dyn92]. f is called *limit function*, and in the special cases $s = 1$ ($s = 2$) - *limit curve* (*surface*). The definition for uniform convergence can be formulated in an epsilon-delta way:

Definition 2.8 *S is uniformly convergent, if for any initial sequence $v^0 \in l_\infty(\mathbb{Z}^s)$, any bounded domain $\Omega \subset \mathbb{R}^s$ and $\epsilon > 0$ there exists $J(v^0, \epsilon, \Omega)$ such that*

$$|(S^j v^0)_\alpha - S^\infty v^0(r^{-j}\alpha)| < \epsilon, \quad j > J(v^0, \epsilon, \Omega), \qquad \alpha \in \mathbb{Z}^s \cap r^j\Omega. \tag{2.3}$$

For shift-invariant subdivision schemes S, it is enough to take $\Omega = [0, 1]^s$, since every bounded domain in $\mathbb{R}^s$ can be covered by a finite union of shifts $\Omega + \beta_i, \beta_i \in \mathbb{Z}^s, i = 1, \dots, N$ of Ω. The shift-invariance guarantees

$$J(v^0, \epsilon, \Omega + \beta_{i_1}) = J(v^0, \epsilon, \Omega + \beta_{i_2}) \qquad \forall i_1, i_2 \in \{1, \dots, N\}.$$

Moreover, if S is local then we can take a finite subsequence of v^0 and work only with it. Note, that in this case for every j, v^j will be a finite sequence, but its length increases monotonically with respect to j.

Definition 2.9 *A subdivision scheme S is called (Lipschitz)* ***stable*** *if, there exists a positive constant C, such that for any $v^0, \tilde{v}^0 \in l_\infty(\mathbb{Z}^s)$*

$$\|S^\infty v^0 - S^\infty \tilde{v}^0\| \le C\|v^0 - \tilde{v}^0\|. \tag{2.4}$$

The equivalent epsilon-delta definition for stability is:

Definition 2.10 *S is stable, if there exists a positive, finite constant C such that for any $\epsilon > 0$, any initial sequence $v^0 \in l_\infty(\mathbb{Z}^s)$, and any bounded domain $\Omega \subset \mathbb{R}^s$ there exists $J(v^0, \epsilon, \Omega)$ with the property: for any $\tilde{v}^0 \in B(v^0, \epsilon/C) = \{v \in l_\infty(\mathbb{Z}^s) : \|v - v^0\| < \epsilon/C\}$*

$$|(S^j v^0)_\alpha - (S^j \tilde{v}^0)_\alpha| < \epsilon \quad j > J(v^0, \epsilon, \Omega), \qquad \alpha \in \mathbb{Z}^s \cap r^j\Omega. \tag{2.5}$$

Note that this is a very strong version of stability. Usually the notion of stability is local, i.e. can be formulated as "small perturbation δ of the initial data leads to a small perturbation ϵ of the derived data". The above definitions are more restrictive, since they are for arbitrary ϵ, not only for a sufficiently small one and the ratio δ/ϵ is uniformly bounded by C. Thus, there are authors who prefer weakened versions (see [CDM03]). In this paper, we will use (2.4), unless something else is specified.

In theory, there are two main types of stability - asymptotic and Liapunov (see [Wig03]). If we think of $(v^0, 0)$ and $(\tilde{v}^0, 0)$ as points in the space $l_\infty(\mathbb{Z}^s) \times \{t \ge 0\}$ then $\{(v^j, j)\}$ and $\{(\tilde{v}^j, j)\}$ are discrete trajectories, and the corresponding definitions are:

Definition 2.11 *(**Liapunov Stability.**) Let $S : l_\infty(\mathbb{Z}^s) \to l_\infty(\mathbb{Z}^s)$ be a subdivision scheme. Let $\mathbf{v} := \{v^0, v^1, \dots\}$ be the discrete trajectory with respect to S, associated to the initial data $v^0 \in l_\infty(\mathbb{Z}^s)$ (i.e., $v^{j+1} = Sv^j, \forall j \in \mathbb{N}$). Then $\mathbf{v}$ is **Liapunov stable**, if for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for any $\tilde{\mathbf{v}} = \{\tilde{v}^0, \tilde{v}^1, \dots\}$ satisfying $\|v^0 - \tilde{v}^0\| < \delta$, $\max_{J \in \mathbb{N}} \|v^J - \tilde{v}^J\| < \epsilon$.*
S is Liapunov stable itself, if all the discrete trajectories with respect to S are Liapunov stable.

Definition 2.12 *(**Asymptotic Stability.**) $\mathbf{v}$ is said to be **asymptotically stable** if it is Liapunov stable and for any other trajectory, $\tilde{\mathbf{v}}$ with respect to S, there exists a constant $b > 0$ such that, if $\|v^0 - \tilde{v}^0\| < b$, then $\lim_{J \to \infty} \|v^J - \tilde{v}^J\| = 0$.*

Note that, since we are dealing with stationary subdivision schemes, our dynamical system is autonomous! A subdivision scheme S which reproduces constants, cannot be asymptotically stable, because if we take $v^0 = \mathbf{1}$ and $\tilde{v}^0 = \mathbf{0}$, then for every $j \geq 0$ the distance between $\{(v^j, j)\}$ and $\{(\tilde{v}^j, j)\}$ is 1. (2.5) looks like Liapunov stability but the main drawback of this interpretation is that, unlike the Liapunov stability, the trajectories of "close" initial points stay in a tube around the initial trajectory only after some finite moment, that depends on the initial sequence and is not uniformly bounded. Again, as with the uniform convergence, if S is shift-invariant it suffices to take $\Omega = [0, 1]^s$, and if S is local - we can work with finite subsequences of v^0 and $\tilde{v}^0$.

2.2 Multiresolution

The general framework for multiresolution representation (MR) of data was presented by Harten in [Har96]. For the purposes of this paper, we will restrict ourselves to the one-dimensional dyadic multiresolution, with $S^j = V^j = l_\infty(2^{-j}\mathbb{Z}) =: l_\infty(\Gamma^j)$, and $\mathcal{F} = BC(\mathbb{R})$ - the space of bounded and continuous real-valued functions on $\mathbb{R}$.

2.2.1 MR of sequences

Let the linear operator R_j^{j-1} and the subdivision rule S_{j-1}^j satisfy

$$R_j^{j-1} : l_\infty(\Gamma^j) \to l_\infty(\Gamma^{j-1}), \tag{2.6a}$$

$$S_{j-1}^j : l_\infty(\Gamma^{j-1}) \to l_\infty(\Gamma^j), \tag{2.6b}$$

$$R_j^{j-1} S_{j-1}^j = I_{j-1}, \tag{2.6c}$$

where I_{j-1} is the identity operator in $l_\infty(\Gamma^{j-1})$. R_j^{j-1} is known as the *restriction operator*, and S_{j-1}^j as the *prediction operator*. Since, we are dealing only with stationary subdivision, we need S_{j-1}^j, and thus R_j^{j-1} (see (2.6c)) not to depend on the level j, i.e., for every $j \in \mathbb{N}$

$$S_{j-1}^j = S, \qquad R_j^{j-1} = R. \tag{2.7}$$

We will keep the lower and upper indexes of the two operators just to indicate on which resolution level we are.

Let $v^J \in l_\infty(\Gamma^J)$. Then, using the restriction operator, we generate the sequence $\{v^j\}_{j=0}^J$:

$$v^j = R_{j+1}^j R_{j+2}^{j+1} \dots R_J^{J-1} v^J \in l_\infty(\Gamma^j). \tag{2.8}$$

For each $1 \le j \le J$ we denote by d^j the *detail*

$$d^j := v^j - S_{j-1}^j v^{j-1}. \tag{2.9}$$

Note that although d^j is a sequence on the j-th level of multiresolution, in the interpolatory dyadic case it is more natural to relate it to the $(j-1)$-st one, since $(d^j)_{2k} = 0, \forall k \in \mathbb{Z}$, and the only "meaningful" entries are the odd ones.

Therefore we can represent any sequence v^J in a unique way as a sequence on the zero level of resolution and a sequence of details on all the intermediate resolution levels.

$$v^J \longleftrightarrow M(v^J) := \{v^0, d^1, \dots, d^J\}.$$

We refer to $M(v^J)$ as the MR of v^J. We will assign an operator M to every multiresolution, namely

$$Mv^j := v^{j+1} = Sv^j + d^{j+1}. \tag{2.10}$$

Unlike subdivision, in multiresolution we have two operators and two processes involved: the *direct* MR transform $(v^J \to M(v^J))$, and the *inverse* MR transform $(M(v^J) \to v^J)$. Therefore, in order to talk about stability of a multiresolution, we have to prove stability for both of the transforms.

Definition 2.13 ***(stability of the direct MR)*** *Let $v^J, \tilde{v}^J \in l_\infty(\Gamma^J)$, and let $M(v^J)$ and $M(\tilde{v}^J)$ be the corresponding MRs. We say that the direct MR transform is* ***stable*** *if there exists C such that*

$$\|v^0 - \tilde{v}^0\| \le C\|v^J - \tilde{v}^J\|, \qquad \|d^j - \tilde{d}^j\| \le C\|v^J - \tilde{v}^J\|, \ \ \forall j = 1, \dots, J. \tag{2.11}$$

Definition 2.14 ***(stability of the inverse MR)*** *Let $v^J, \tilde{v}^J \in l_\infty(\Gamma^J)$, and let $M(v^J)$ and $M(\tilde{v}^J)$ be the corresponding MRs. We say that the inverse MR transform is* ***stable*** *if there exists C such that*

$$\|v^J - \tilde{v}^J\| \le C\big(\|v^0 - \tilde{v}^0\| + \sum_{j=1}^J \|d^j - \tilde{d}^j\|\big). \tag{2.12}$$

2.2.2 MR of functions

Analogously, to the MR of discrete data, one can define a multiresolution representation of continuous data. Therefore, for any $j \in N$ Harten introduced the linear operator $R_j : BC(\mathbb{R}) \to l_\infty(\Gamma^j)$ and the operator $S_j^\infty : l_\infty(\Gamma^j) \to BC(\mathbb{R})$ such that

$$R_j S_j^\infty = I_j. \tag{2.13}$$

R_j is called a *discretization* operator, and S_j^∞ a *reconstruction* operator.

A classical example about the relation between Section 2.2.1 and Section 2.2.2 is the use of an interpolatory, uniformly convergent subdivision rule S as the prediction operator ($S^j_{j-1} = S, \forall j \in \mathbb{N}$). Then, a possible choice for the restriction operator is

$$(R_j^{j-1} v^j)_k = v^j_{2k}, \qquad k \in \mathbb{Z}, \ \ j \in \mathbb{N}. \tag{2.14}$$

We can take the discretization operator to be the projection of f on Γ^j

$$(R_j f)_k = f(2^{-j}k), \qquad f \in BC(\mathbb{R}), \ \ , k \in \mathbb{Z}, \ \ j \in \mathbb{N}, \tag{2.15}$$

and the reconstruction operator to be the limit action of S

$$S_j^\infty v^j = S^\infty v^0. \tag{2.16}$$

We refer to the multiresolution operator defined by this algorithm, as the *multiresolution associated to S*. Most of the examples of multiresolution operators that we consider in this paper are exactly of this type.

To prove stability of a multiresolution M associated to an interpolatory subdivision scheme S, we need to prove stability of the direct and the inverse MR. Letting all the details be zero in (2.12), we obtain that a necessary condition for stability of the inverse MR is the stability of the subdivision scheme S, itself. In this particular case of M this condition is enough for proving stability of the direct MR.

Lemma 2.15 *Let S be stationary, interpolatory, stable subdivision scheme. Let M be the multiresolution associated to S. Then the direct MR is stable!*

Proof. First, note that since S is interpolatory, v^i is a subsequence of v^j for every $j \geq i$. Therefore, $\|v^i - \tilde{v}^i\| \leq \|v^j - \tilde{v}^j\|$. In particular $\|v^0 - \tilde{v}^0\| \leq \|v^J - \tilde{v}^J\|$ for every $J \in \mathbb{N}$ and we only have to prove that

$$\|d^j - \tilde{d}^j\| \leq C\|v^j - \tilde{v}^j\| \qquad \forall j \in \mathbb{N},$$

where the constant C must be uniformly bounded.

$$\|v^j - \tilde{v}^j\| = \|Sv^{j-1} + d^j - S\tilde{v}^{j-1} - \tilde{d}^j\| \geq \|d^j - \tilde{d}^j\| - \|Sv^{j-1} - S\tilde{v}^{j-1}\|.$$

Now using that S is stable and that $(v^{j-1} - \tilde{v}^{j-1})$ is a subsequence of $(v^j - \tilde{v}^j)$, we derive

$$\|v^j - \tilde{v}^j\| \geq \|d^j - \tilde{d}^j\| - C_1\|v^{j-1} - \tilde{v}^{j-1}\| \geq \|d^j - \tilde{d}^j\| - C_1\|v^j - \tilde{v}^j\|.$$

Since S is stationary, C_1 and thus $C := C_1 + 1$ do not depend on j and the lemma is proved. $\square$

Corollary 2.16 *Let S be interpolatory subdivision rule. The multiresolution M associated to S is stable if and only if there exists a constant C such that for any $J \in \mathbb{N}$*

$$\|v^J - \tilde{v}^J\| \leq C\big(\|v^0 - \tilde{v}^0\| + \sum_{j=1}^{J} \|d^j - \tilde{d}^j\|\big). \tag{2.17}$$

3 Survey of the existing literature

In the context of image (curve, surface) representation as well as in the context of numerically computing weak solutions of nonlinear conservation laws, piecewise smooth data with jump discontinuities appear. Due to the Gibbs phenomena, a linear multiresolution operator cannot simultaneously localize these discontinuities and provide smooth approximation to the initial function away from the singularities. Another big problem is that convexity preserving linear subdivision schemes are at most C^1 smooth even when the data is strictly convex. These drawbacks motivates the investigation of non-linear subdivision schemes and their usage in a nonlinear multiresolution transforms.

Several subdivision schemes have been introduced and analyzed so far. Harten, Engquist, Osher and Chakravarthy [HEOC87] proposed the essentially non-oscillatory (ENO) subdivision scheme, which uses not only central cubic interpolation like S_c but also left and right cubic ones (i.e., the interpolating nodes are shifted by one once to the left, and once to the right). If we denote by p_k^l, p_k^c, and p_k^r the corresponding interpolating polynomials, then for v_{2k+1}^{j+1} we use the value at $2^{-(j+1)}(2k+1)$ of the least oscillatory one (fig. 2a). This method is uniformly convergent [CDM03], behaves like S_c away from singularities, and prevents the Gibbs phenomena around them. Unfortunately, small perturbations of data may lead to changing the interpolating polynomial, and thus ENO is unstable. Therefore, the weighted ENO (WENO) subdivision scheme was suggested [LOC94], where instead of taking only one of the interpolating polynomials for imputation, one uses a convex combination

$$p_k = \alpha_1 p_k^l + \alpha_2 p_k^c + \alpha_3 p_k^r, \qquad \alpha_1 + \alpha_2 + \alpha_3 = 1; \;\; \alpha_1, \alpha_2, \alpha_3 \geq 0$$

of all the three polynomials with data-dependent weights $\{\alpha_i\}_1^3$. Extensions of these schemes by tensor-product techniques to 2D edge-adaptive image representation have been considered and numerically investigated in [AACD02], [AAC$^+$01], [CDSB97], [CDSB00]. Both ENO and WENO are built on linear subdivision rules, but the data-dependence of these rules is what makes the whole process nonlinear. In [CDM03] Cohen, Dyn, and Matei analyze the convergence, smoothness, and stability of a larger class of so called quasilinear datadependent subdivision schemes, defined by an operator-valued map Φ, such that

$$Sv = \Phi(v)v, \qquad \forall v \in l_\infty(\mathbb{Z}), \tag{3.1}$$

and $\Phi(v) : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ is linear for every v. Using the developed theory, the authors prove stability of the WENO scheme. However, due to the form of the weights $\{\alpha_i\}_1^3$, the constant C from (1.1) heavily depend on a fixed parameter ϵ. The question what the asymptotical behavior of WENO is when $\epsilon \to 0$ as well as the one about stability of the associated multiresolution are still open.

The idea behind the so far introduced schemes is: take a set of "smooth enough" linear subdivision rules, choose one of them for smooth data and always apply it in such occasions, and use adaptivity around discontinuities. Thus the limit function has the same smoothness as the fixed subdivision rule away from singularities, and lacks smoothness but removes the Gibs phenomena around them. An "opposite" approach is to use the simplest linear subdivision rule - midpoint linear interpolation S_L, defined by $(S_L v)_{2i} = v_i$, $(S_L v)_{2i+1} = \frac{v_i + v_{i+1}}{2}$ and to perturb it away from discontinuities so that to increase the smoothness of the limit. Kuijt and van Damme introduced

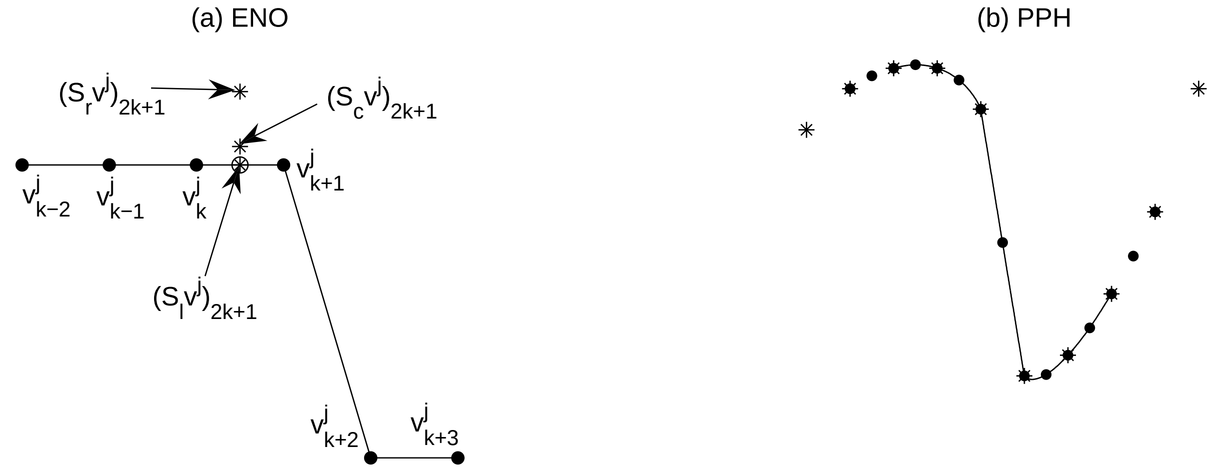

Figure 2: (a) ENO algorithm: the three candidates for imputation, and the less oscillatory choice; (b) one step of PPH and the limit function.

two classes of nonlinear subdivision schemes, based on that idea, which in addition are monotonicity [KvD99] and convexity [KvD97] preserving. The most exploited member of the second class is the piecewise polynomial harmonic (PPH) scheme, introduced also by Floater and Micchelli in [FM98], which is interpolatory and dyadic with

$$(S_{PPH}v)_{2k+1} = \frac{v_k + v_{k+1}}{2} - \frac{1}{8} H(\triangle^2 v_{k-1}, \triangle^2 v_k), \tag{3.2}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is the harmonic mean defined by

$$H(x,y) = \begin{cases} \frac{2xy}{x+y}, & xy > 0 \\ 0, & otherwise \end{cases}.$$

PPH is a second order perturbation of S_L, i.e., $S_{PPH}v = S_L v + F(\triangle^2 v)$, where F is a nonlinear operator. In [AL05] stability of PPH and the corresponding multiresolution is proven. The idea of this proof stays in the core of our stability theorem. Proof for stability of a large subclass of the schemes proposed in [KvD97], including PPH, can be found in [KvD98b]. In [ADLT06] a bivariate scheme, based on tensor product on PPH is introduced and analyzed in terms of convergence and stability.

Experimentations with the ENO method have indicated several practical drawbacks, therefore Serna and Marquina [SM04] introduce a new class of reconstruction procedures, the Power ENO methods, which are based on an extended class of limiters

$$\text{power}_p(x,y) = \frac{(x+y)}{2}\Big(1 - \Big|\frac{x-y}{x+y}\Big|^p\Big),$$

defined as a correction of classical ENO methods. This class contains both WENO and PPH (take $p = 2$ in the power$_p$ mean), reduces smearing near discontinuities, and provides good resolution of corners and local extrema. Moreover a new weighted ENO method (Weighted Power-ENO5) is proposed, which has the highest possible order of polynomial reproduction - six. The convergence of these methods and their extension to 2D are subject of [DL]. Stability is an open question.

A family of nonlinear adaptive 4-point interpolatory, data-dependent subdivision schemes for improving image reconstruction is introduced in [MDL04]. These schemes are proven to uniformly converge and are experimentally shown to be C^1. Stability is, again, an open question.

Normal meshes and multiresolution for curves and surfaces is a recent concept in efficient geometry representation using local data-dependent coordinate systems adapted to tangential and normal directions. Here the corresponding subdivision scheme is usually linear but the details are not linearly added, i.e., instead of having $Mv = Sv + d$ we have $Mv = F(v, d)$, where F is linear in v and nonlinear in d. More precisely,

$$(Mv^j)_{2k+1} = (Sv^j)_{2k+1} + (\sigma_j d^j \mathbf{n}_j)_k, \qquad (Mv^j)_{2k} = v^j_k, \qquad v^j = (x^j, y^j) \in l_\infty(\mathbb{Z}) \times l_\infty(\mathbb{Z})$$

where $\mathbf{n}_j = \frac{\triangle v^{j,\perp}}{\|\triangle v^j\|_2} := \frac{(-\triangle y^j, \triangle x^j)}{\|\triangle v^j\|_2}$ is the corresponding normal vector (fig. 3a). This idea was originally introduced by Guskov and others in [GVSS00]. [DRS04] studies properties like regularity, convergence, and stability of normal multiresolution analysis for curves. The central idea of the paper is to study normal approximation as a perturbation of a linear subdivision scheme. This works only under additional assumptions on the smoothness of the initial curve Γ, thus only $C^\beta(\mathbb{R}^2), \beta > 1$ curves are considered. Furthermore, the authors always assume that Γ can be parameterized by its x-coordinate and interpret the normal multiresolution as a nonregular scalar-valued multiresolution on the grid x^j. To avoid difficulties, they assume that the new nodes are “well placed”, i.e., starting with a monotonic sequence x^0, to have monotonic x^j's for all $j > 0$, and the points of x^j not to cluster (both this conditions depend on the parametrization of Γ, and one can usually obtain them via rotation of the coordinate system). Under these assumptions the authors manage to prove uniform exponential convergence, regularity (that depends on the regularity of S and the smoothness of Γ), geometrical decay of the details, and finally stability of the normal multiresolution. Moreover, they produce a counterexample that the results above do not hold for initial curves $\Gamma \in C^\beta, \beta \leq 1$. The question about normal surfaces and how normal meshes work for less smooth spaces, particularly spaces that are used to model natural images such as BV, and the Besov space $B_{1,1}$ remains open. The other uninvestigated direction is using non-interpolatory linear schemes as a subdivision rule.

[KG03] implements the normal “remeshing” technique, based on the Butterfly subdivision scheme [DLG90] introduced in [GVSS00] for irregular highly detailed meshes and implements it in the progressive geometry coding described in [KSS00]. This paper is an attempt for generalizing the theory in [DRS04] from the curve to the surface case. The authors use both the Butterfly scheme again, and the Loop [Loo87] scheme for the wavelet transform. Measuring the performance of the above coding gives better compression rate of normal meshes obtained using Butterfly wavelets than the Loop ones, and a better compression rate of normal meshes versus other remeshes when the Loop wavelets are used. The first result is intuitive, but the second one is very interesting and is not theoretically proven, yet. The smoothness of the normal mesh parametrization is expected, but is also an open question. Another open problem is whether using non-linear optimization approach rather than the Butterfly scheme for normal remeshing will significantly decrease the percentage of non-normal coefficients and thus will be worth investigating. (The non-normal coefficients come from locations where the piercing did not find any “valid” intersection point with the original surface, and for adaptive semi-regular meshes, produced by the Butterfly optimization and having the same number of vertices as the original irregular mesh their number is below 10%.)

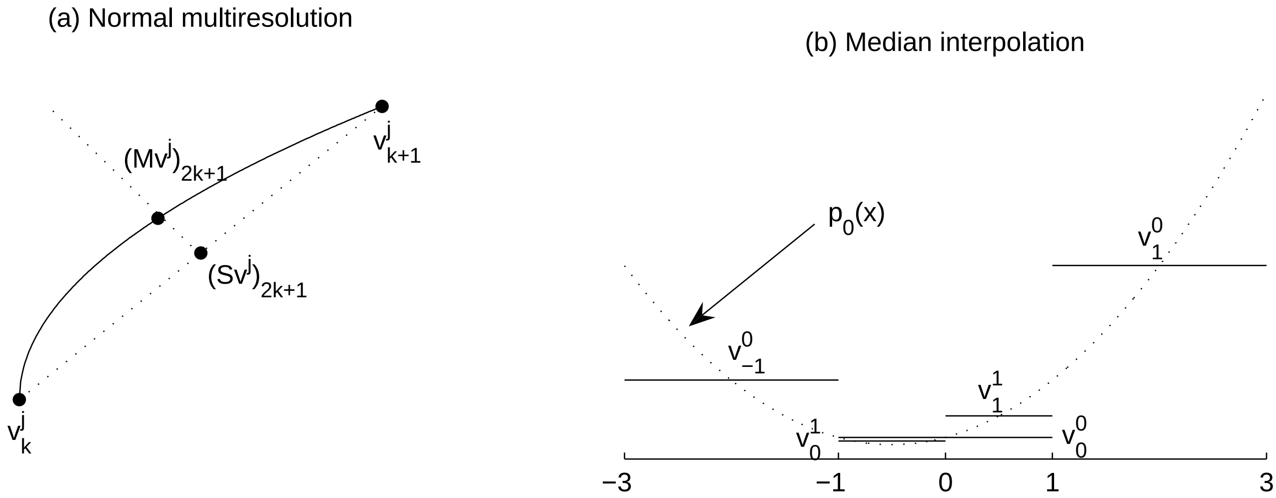

Figure 3: (a) Normal multiresolution using the midpoint-interpolating subdivision scheme. (b) One step of the median-interpolating subdivision scheme

Another possible direction of generalization and improving the progressive geometry compression is allowing irregular refinements, as well. Such kind of approach is investigated in [GKSS02], where each step of a regular refinement is followed by an irregular one. The irregular operations are limited and they are as many as needed for topology changes, feature alignment, and stretching resolution. The main problem of the regular multiresolution is that "spiky" features lead to a stretched map, which may cause bad aspect ratio polygons, poor approximation, and numerical problems. On the other hand, the irregular algorithms are more complex for multiresolution, smoothing, compression, editing, etc. The above construction combines the advantages of both the refinements, and "controls" the disadvantages. All the results in the paper are experimental, which leaves room for research. Moreover, representing high topological complexity models such as isosurfaces from medical imaging or scientific computing, and approximating dynamically changing, topologically complex geometries does not give good results with the novel approach, so further generalizations are expected.

In the signal processing, linear multiresolution is a promising approach for successfully removing Gaussian noise. Unfortunately, the linear case does not work with strongly non-Gaussian noise, examples of which are typical in analogue telephony, radar signal processing, and laser radar imaging, since classical statistical models show that maximum likelihood estimators are often linear in the Gaussian case, but highly nonlinear when Cauchy noise appears. The remedy is to use robust estimators. In [DY00] Donoho and Yu introduce a class of nonlinear, triadic subdivision schemes based on median interpolation, one of which, the quadratic median-interpolating scheme $S_{med,3}$ (MI), allows a closed-form representation and is the main object of investigation. Its dyadic version S_{med} [Osw04] is computationally easier to deal with, and although it does not have the same denoising properties as the original triadic scheme, it can be used for testing theoretical results.

Let f be real-valued continuous function on a bounded interval I with positive Lebesgue measure ($m(I) > 0$). Then the *median* of f on I is defined by

$$med(f; I) := \sup\left\{\alpha : m(\{x : f(x) < \alpha\}) \leq \frac{1}{2}m(I)\right\}. \tag{3.3}$$

$S_{med,3}$ uses the coarse-level information on three consecutive intervals to construct the quadratic polynomial p that has the same medians in this intervals as the given data, and use its medians on the intervals, produced from the central interval by the grid refinement, to impute data on the next level (fig. 3b). This scheme is "closely" related to two linear schemes: the cubic midpoint-interpolating scheme $S_{mid,3}$ and the average interpolating scheme. In [DY00] uniform convergence and Hölder-α regularity for $S_{med,3}$ is proven with $\alpha > 0.0997$. Oswald [Osw04] improves the lower bound to $\alpha > 0.8510$, before Xie and Yu [XY05] to prove $\alpha > 1 - \epsilon$ which is exactly the critical Hölder exponent for $S_{mid,3}$, as conjectured in [DY00]. The question about stability remains open, and its proof is the first goal in our research. For S_{med} we can prove stability, up to one case for which strong numerical evidences are provided. After theoretically proving this remaining case, we want to adjust the computations to the triadic setting. Using MI, Donoho and Yu [DY00] develop a median-interpolating pyramid transform (MIPT) and analyze the transform coefficients. There are several distinguishable things in this construction. First, unlike in the types of the geometry-driven applications, it is better to associate a piece-wise constant function to the sequence $v^k \in l_\infty(\mathbb{Z})$ rather than the linearly interpolating one (fig 3b), i.e.,

$$g_j(\cdot) = \sum_{k \in \mathbb{Z}} v_j^k 1_{I_{j,k}}(\cdot), I_{j,k} = [k3^{-j}, (k+1)3^{-j}).$$

Second, $S_{med,3}$ is triadic and non-interpolatory, thus the transform is expensive. Finally, the restriction operator (taking the discrete median of the values at the three fine-level intervals) is nonlinear. Despite all these "oddities", the median-interpolating pyramid transform satisfies the typical properties of a wavelet transform, such as coefficient localization, coefficient decay, and Gaussian-noise removal. Moreover, it removes Cauchy noise with the same efficiency as the Gaussian one. Stability of MIPT is crucial for applications, and is not proven yet. Our experience with the dyadic case shows that without additional assumptions on the smoothness of the initial function f (as in normal multiresolution) the transform is not stable.

In [GH00a] and [GH00b] Goutsias and Heijmans present a general theory for constructing linear as well as nonlinear pyramid and wavelet decomposition schemes for signal analysis and synthesis. The main assumptions correspond to the Harten's approach and are: "perfect reconstruction", which states that subdivision followed by restriction returns the original signal, and that the grids are nested. The proposed theory unites different tools for constructing multiresolution signal decomposition schemes, such as pyramids, wavelets, morphological skeletons and granulometries, and gives a complete characterization of restriction and subdivision operators between two adjacent levels.

The necessity of considering manifold-valued data arises from real-life examples like: headings, orientations, rigid motions, deformation tensors, distance matrices, projections, subspaces, etc. If the control points of a subdivision are restricted to a certain manifold, surface, or a Lie group, then, due to these restrictions, even when the original subdivision scheme is linear, its modification becomes nonlinear. The questions about how to modify the linear subdivision scheme and which of its properties can be inherited by the associated manifold-valued scheme are discussed in [WD05], [Wal06], [WP06], [WNYG07], [GW07], [Gro06]. The analysis of the latter is based on its *proximity* to the linear scheme it is derived from. This general principle has been used before, e.g. in the analysis of non-stationary linear schemes in [DL95], and nonlinear schemes in [DRS04], [XY05], [XY06].

In [URDS$^+$05] a generalized wavelet analysis on a Manifold-valued data is introduced. It is analogous to the wavelet analysis in $\mathbb{R}$, but in the generalized wavelet transform, like in normal multiresolution, there is an important structural distinction between the coarse-scale information, which belong to the manifold M and the fine scale information (i.e., the details) which consists of tangent vectors to M. The main idea of the new approach is to project the manifold-valued data onto a tangent space $T_{p_0}(M)$ via the *Exp* map, where $p_0 \in M$ is suitably chosen, then to apply the corresponding real-valued linear subdivision scheme, and finally to go back on M via the *Log* map. This procedure works only locally, i.e., when the data $p(k)$ (this is the part of p which is necessary for obtaining p_{2k+1}), and the imputed point p_{2k+1} can be mapped onto a single tangent plane. Due to different applications, both "point-concentrated" (based on S_c) and "interval-concentrated" (based on Average-Interpolating) pyramids are considered. This approach is shown to work well on various test and real-life examples, but no mathematical proofs are given. It is conjectured that the smoothness of the new subdivision schemes coincide with the smoothness of the underlaying linear ones.

In [WD05] proximity conditions are defined, and general theorems for convergence and C^1 smoothness of a nonlinear scheme T that satisfy proximity condition with an affine-invariant, interpolatory, linear, univariate subdivision scheme S are proven. Then, since S can be expressed in terms of repeated affine averages

$$\mathrm{av}_\alpha(x, y) := (1-\alpha)x + \alpha y, \quad \alpha \in \mathbb{R},$$

the authors suggest in order to restrict the data to a manifold or a group either to substitute the affine averages by geodesic ones or to project the affine averages onto the manifold or the group. Finally, it is proven that (under some local restrictions) for a given S all the analogous geodesic schemes (both for a surface and for a matrix group of constant velocity), and all the analogous projecting schemes fulfil proximity conditions, and hence inherit the C^0 and C^1 smoothness of S. [Wal06] is a continuation of [WD05], where the proximity analysis is extended and applied for $C^k, k \in \mathbb{N}$ smoothness. Again, only the univariate case is considered, and the geodesic and projective constructions from the previous article are shown to inherit C^2 smoothness, as well. [WNYG07] deals with a particular log-exponential geodesic analogue of a univariate, linear, dyadic subdivision in Lie groups. This analogue is shown to satisfy the generalized proximity conditions from [Wal06] with $k = 2$, and thus to be C^2 smooth. [GW07] studies an alternative log-exponential analogue, which allows arbitrary dilation factors, and is better suited for generalization to the multivariate setting. The authors prove that the proximity conditions are satisfied and give sufficient conditions for convergence, C^1 and C^2 smoothness of the introduced schemes. Finally, [Gro06] extends the theory from [WD05] to the multivariate case. It is proven that the proximity conditions are satisfied for a large class of nonlinear multivariate schemes, namely the bivariate geodesic-average and projection-average analogues (defined in [WD05]), the "closest point projection" as well as the Log-Exp analogue (defined in [GW07]), and thus (under some local restrictions on the initial data) the above proximity schemes, defined on an arbitrary abstract Riemannian manifold or an arbitrary abstract Lie group, have the same smoothness (up to C^1) as the underlaying linear subdivision scheme S.

4 Linear subdivision

4.1 Basic representations

Let us start with an example (see [Dyn92]). The centered *B-spline* B_n of degree n over $\mathbb{Z}$ in $\mathbb{R}$ (or arbitrary regular grid Γ) is a real-valued, $(n-1)$-times differentiable, non-negative piecewise polynomial function of degree n compactly supported on $[-\lfloor\frac{n+1}{2}\rfloor, \lceil\frac{n+1}{2}\rceil]$. Let

$$S_{n,\mathbb{Z}} = \{f : f \in C^{n-1}, f|_{(k,k+1)} \in \Pi_n(\mathbb{R}), k \in \mathbb{Z}\}$$

be the space of all splines of degree n over $\mathbb{Z}$. The union of all integral shifts of B_n $\{B_n(x-k) : k \in \mathbb{Z}\}$ (or d-shifts in general, where d is the step-size of Γ) form a basis of $S_{n,\mathbb{Z}}$ ($S_{n,\Gamma}$) and partition of unity of $\mathbb{R}$:

$$\sum_{k\in\mathbb{Z}} B_n(x-k) = 1, \qquad x \in \mathbb{R}. \tag{4.1}$$

The easiest way to define $B_n(x)$ is by recursion. Take $B_0(x) = \chi_{[0,1]}(x)$ - a piecewise constant function, that is one in $[0,1]$ and zero otherwise. Then

$$B_{n+1}(x) = \int_{x-\epsilon_n}^{x-\epsilon_n+1} B_n(t)dt, \qquad \epsilon_n = n \bmod 2.$$

Let $f \in S_{n,\mathbb{Z}}$. Then

$$f(x) = \sum_{k\in\mathbb{Z}} v_k^0 B_n(x-k), \tag{4.2}$$

where $v_k^0 \in \mathbb{R}, k \in \mathbb{Z}$ are uniquely determined.

But for every $j \in \mathbb{Z}$, $S_{n,2^j\mathbb{Z}} \subset S_{n,\mathbb{Z}}$. Hence, $f(x) \in S_{n,2^j\mathbb{Z}}$, as well, where $\{B_n(2^jx-k) : k \in \mathbb{Z}\}$ form a basis. Thus

$$f(x) = \sum_{k\in\mathbb{Z}} v_k^j B_n(2^jx-k), \tag{4.3}$$

with unique real coordinates v_k^j. In the special case, when $f(x) = B_n(x)$ and $j = 1$ we obtain:

$$B_n(x) = \sum_{l\in\mathbb{Z}} a_{l,n} B_n(2x-l). \tag{4.4}$$

Note that, due to the compact support of the B-spline, only finitely many $a_{l,n}$'s are nonzero. Therefore the sum in (4.4) is finite.

Now, combining (4.3) and (4.4) we derive

$$f(x) = \sum_{l\in\mathbb{Z}} v_l^j B_n(2^jx-l) = \sum_{l\in\mathbb{Z}} v_l^j \sum_{k\in\mathbb{Z}} a_{k,n} B_n(2^{j+1}x-2l-k) = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} a_{k-2l,n} v_l^j B_n(2^{j+1}x-k).$$

Hence,

$$v_k^{j+1} = \sum_{l\in\mathbb{Z}} a_{k-2l,n} v_l^j, \qquad \forall k \in \mathbb{Z}. \tag{4.5}$$

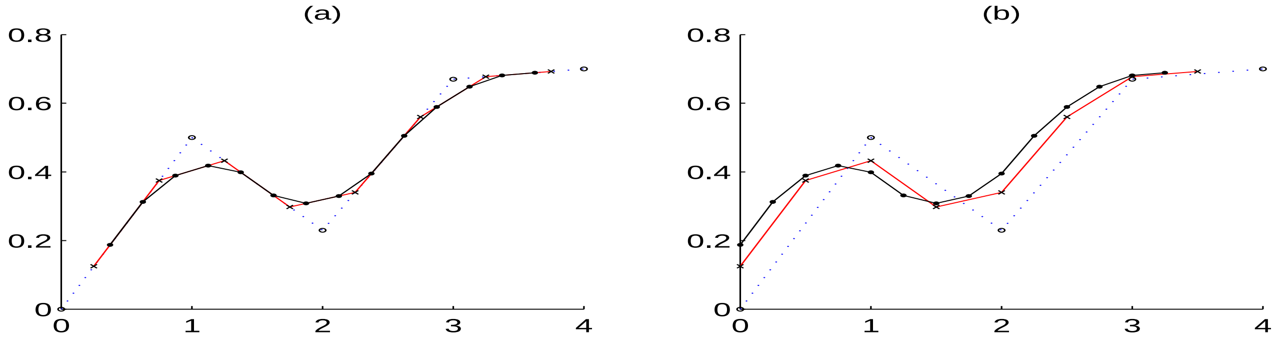

Figure 4: Chaikin's algorithm: (a) practical application; (b) theoretical setup.

It is well known (see [LR80]) that letting j tend to infinity, $\{(2^{-j}k, v_k^j)\}$ converges uniformly to f. Thus, using (4.4) and (4.5), we can construct a uniformly convergent, linear subdivision scheme $S : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ such that

$$(Sv)_k = \sum_{l\in\mathbb{Z}} a_{k-2l,n} v_l \qquad \forall k \in \mathbb{Z}. \tag{4.6}$$

In the literature, S is known as *uniform B-spline subdivision scheme*.

Note that we did everything "backwards". We started with the limit curve, "projected" it on the grids $2^{-j}\mathbb{Z}$, and then "obtained" the linear subdivision rule.

If we take $n = 1$ and do explicitly all the above calculations, we get:

$$(S_L v)_{2k} = v_k, \qquad (S_L v)_{2k+1} = \frac{v_k + v_{k+1}}{2}. \tag{4.7}$$

This scheme is *interpolatory*, since all the points at all levels stay on the limit curve and is known as: the degree 1 centered Lagrange interpolation. The limit curve f coincides with f^j for all $j \in \mathbb{Z}$.

$n = 2$ gives:

$$(Sv)_{2k} = \frac{3}{4}v_k + \frac{1}{4}v_{k-1}, \qquad (Sv)_{2k+1} = \frac{3}{4}v_k + \frac{1}{4}v_{k+1}. \tag{4.8}$$

This scheme is known as Chaikin's algorithm ([Cha74]) and is example of a non-interpolatory subdivision scheme.
Chaikin's algorithm provides an example that the theoretical setup and the practical application of a subdivision scheme could sometimes differ. This is due to the different interpretations of the data. In theory, we think of v^j as an $\mathbb{R}$ data over the regular grid $2^{-j}\mathbb{Z}$ (fig. 4b), while in practice, we think of v^j as a set of *control points* in $\mathbb{R}^2$ (fig. 4a). This ambiguity has no impact on mathematical properties of the subdivision scheme, such as convergence, stability and smoothness, since for any $v^0 \in l_\infty(\mathbb{Z})$, at any level $j \in \mathbb{Z}$ f_a^j is a shift of f_b^j by $d_j = \sum_{i=0}^{j} 2^{-(1+j)}$ and, thus, this is true for the two limit functions, as well!

There are several representations of a local, linear subdivision scheme S that are used:

1) By a $2s$-infinite matrix $S = (s_{\alpha,\beta}), \alpha, \beta \in \mathbb{Z}^s$.

2) By its *mask* $\mathbf{a} \in l_\infty(\mathbb{Z}^s)$.

3) By a finite collection of linear rules.

4) By its action on the Kronecker delta.

Since S is a linear operator from $l_\infty(\mathbb{Z}^s)$ to itself, we can think of it as a $2s$-infinite matrix $S = (s_{\alpha,\beta})$, such that for any $v^0 \in l_\infty(\mathbb{Z}^s)$

$$(Sv^0)_\alpha = \sum_{\beta \in \mathbb{Z}^s} s_{\alpha,\beta} v^0_\beta. \tag{4.9}$$

The locality of the subdivision scheme guarantees that the matrix is very sparse, i.e. it gives

$$s_{\alpha,\beta} = 0, \qquad |\alpha - r\beta| > B, \tag{4.10}$$

where r is the dilation factor of S and B is as in definition 2.1. For example, for the Chaikin's algorithm we have: $s = 1, r = B = 2$, and the non-zero entries of the matrix S are

$$s_{2k,k-1} = 1/4, \qquad s_{2k,k} = 3/4, \qquad s_{2k+1,k} = 3/4, \qquad s_{2k+1,k+1} = 1/4, \qquad k \in \mathbb{Z}.$$

$$S = \begin{pmatrix} \ddots & & & & & & & \iddots \\ & 0 & 1/4 & 3/4 & 0 & 0 & 0 & \\ & 0 & 0 & 3/4 & 1/4 & 0 & 0 & \\ & 0 & 0 & 1/4 & 3/4 & 0 & 0 & \\ & 0 & 0 & 0 & 3/4 & 1/4 & 0 & \\ & 0 & 0 & 0 & 1/4 & 3/4 & 0 & \\ \iddots & & & & & & & \ddots \end{pmatrix}.$$

Using $2s$-infinite matrix is not comfortable, since the matrix is extremely sparse, infinite in all directions, and it doesn't use the shift-invariance property of the subdivision scheme. Therefore, it is more convenient to use the mask $\mathbf{a} \in l_\infty(\mathbb{Z}^s)$ of the subdivision scheme S for defining its action. In other words

$$(Sv)_\alpha = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha - r\beta} v_\beta \qquad v \in l_\infty(\mathbb{Z}^s), \tag{4.11}$$

where, again due to locality, $a_\alpha = 0$ for $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_s| > B$. We will use the notion "mask" interchangeably for both $\mathbf{a}$ and its minimal finite subsequence, that contains all the non-zero entries of $\mathbf{a}$. Obviously, $s_{\alpha,\beta} = a_{\alpha - r\beta}$. For the Chaikin's algorithm the mask is

$$\mathbf{a} = (a_{-1}, a_0, a_1, a_2) = (1/4, 3/4, 3/4, 1/4).$$

Using a system of linear rules

$$(Sv)_{r\alpha+\gamma} = \sum_{\beta \in \mathbb{Z}^s} a_{r\alpha+\gamma-r\beta} v_\beta = \sum_{\beta \in \mathbb{Z}^s} a_{\gamma+r\beta} v_{\alpha-\beta}, \qquad \gamma \in E_s, \;\; \alpha \in \mathbb{Z}^s, \;\; v \in l_\infty(\mathbb{Z}^s) \tag{4.12}$$

for defining S, helps visualizing and understanding the geometry of the subdivision scheme. Here, $E_s = \{\gamma : \gamma_i \in \{0, 1, \dots, r_i - 1\}, i = 1, 2, \dots, s\}$. For the Chaikin's algorithm this representation is (4.8).

Let $\delta = \delta_{\alpha,0}$ be the Kronecker delta. Here $0 \in \mathbb{Z}^s$, and $\delta_{\alpha,\beta}$ is the Kronecker symbol. Let S be linear and local. Then, if we know the action of S on δ, we know the action of S on every $v^0 \in l_\infty(\mathbb{Z}^s)$:

$$v^1_\alpha = \sum_{\beta \in \mathbb{Z}^s} v^0_\beta (S\delta)_{\alpha-\beta}. \tag{4.13}$$

The sum is finite, since S is local.
When S is uniformly convergent $\varphi(x) = S^\infty \delta$ is called the *S-refinable function*, and we have

$$S^\infty v^0 = \sum_{\alpha \in \mathbb{Z}^s} v^0_\alpha \varphi(\cdot - \alpha), \qquad \forall v^0 \in l_\infty(\mathbb{Z}^s). \tag{4.14}$$

Note that $\varphi(x)$ has compact support, since S is local and uniform convergent (for the Chaikin's algorithm $\varphi = B_2$). Every uniform convergent, linear subdivision scheme S should reproduce constants (see [CDM91]), so if we take $v^0 = \mathbf{1}$ in (4.14), we see that the integral shifts of φ form a partition of unity in $\mathbb{R}^s$:

$$\sum_{\alpha \in \mathbb{Z}^s} \varphi(x - \alpha) = 1, \qquad x \in \mathbb{R}^s. \tag{4.15}$$

The above two properties of φ give $l_\infty(\mathbb{Z}^s)$-linear independence of $\Phi = \{\varphi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$, i.e., if a (not necessary finite) linear combination of elements of Φ vanishes in $\mathbb{R}^s$, this linear combination must be the trivial one.
Finally, there is a relation between the mask and the refinable function for a uniform convergent scheme S. Plugging $v = \delta$ in (4.11) gives $(S\delta)_\alpha = a_\alpha$. Now, using (4.14) and the fact that δ on $\mathbb{Z}^s$ and $S\delta$ on $r^{-1}\mathbb{Z}^s$ have the same limit functions, we derive

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \varphi(rx - \alpha), \qquad x \in \mathbb{R}^s. \tag{4.16}$$

By induction on $j \in \mathbb{Z}$, we get

$$\sum_{\alpha \in \mathbb{Z}^s} v^j_\alpha \varphi(r^k x - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} v^0_\alpha \varphi(x - \alpha), \qquad \forall v^0 \in l_\infty(\mathbb{Z}^s). \tag{4.17}$$

Therefore, as in the case of B-splines, we can interpret v^j as the coordinates of the limit function $S^\infty v^0$ in span $\{\varphi(r^j x - \alpha) : \alpha \in \mathbb{Z}^s\}$. The representation of a uniformly convergent scheme S via its refinable function is somehow dual to the previous three approaches. Instead of starting from the coarsest grid $\mathbb{Z}^s$ and "refining" the initial data ((4.9), (4.11), (4.12)), one begins with the limit function and "projects" it on the corresponding grid ((4.17)).

We showed that to every uniformly convergent subdivision scheme S we can assign a unique refinable function φ. The converse is also true (see proposition 2.3 in [CDM91]) but only if φ satisfies (4.15), (4.16) with finite mask $\mathbf{a}$, and the "stability hypothesis"

$$\|v^0\| \le c \| \sum_{\alpha \in \mathbb{Z}^s} v^0_\alpha \varphi(x - \alpha) \| \tag{4.18}$$

for any $v^0 \in l_\infty(\mathbb{Z}^s)$. In our case (4.18) is not much stronger than Φ being $l_\infty(\mathbb{Z}^s)$-linearly independent and we can use the latter restriction, but the situation is different with $L_p(\mathbb{R}^s), 1 \leq p < \infty$. For example, the compact support of φ guarantees the $l_2(\mathbb{Z}^s)$-linear independence, but L_2-stability holds if and only if (see [CDM91])

$$\{x : \hat{\varphi}(x + 2\pi\alpha) = 0 : \alpha \in \mathbb{Z}^s\} = \emptyset$$

with $\hat{\varphi}$ being the Fourier transform of φ. For deeper analysis in this direction, one can see [JW93].

Hence, there is a one to one correspondence between the class of uniformly convergent subdivision schemes $S : l_\infty(\mathbb{Z}^s) \to l_\infty(\mathbb{Z}^s)$ and $SC_0(\mathbb{R}^s)/_\sim$: the class $SC_0(\mathbb{R}^s)$ of compactly supported, continuous functions $\varphi : \mathbb{R}^s \to \mathbb{R}$ that satisfy (4.15), (4.16) with finite mask $\mathbf{a}$, and (4.18), quotioned by the equivalence relation $\sim$, where

$$\varphi \sim \psi \iff \exists \alpha \in \mathbb{Z}^s : \varphi = \psi(\cdot - \alpha).$$

The above bijection allows to attack the problems, concerning uniformly convergent schemes from two opposite sides, and plays an important role in most of the proofs!

4.2 Stability analysis

In the linear case, the question about stability is completely solved (see [CDM91]). Here, we will repeat the main results.

Lemma 4.1 *A local, linear subdivision scheme S is stable, if and only if it is uniformly convergent.*

Proof. The definition (2.4) explicitly implies that S is uniformly convergent.
Now, let S be uniformly convergent, and let φ be its refinable function. Denote by C the L_∞-norm of φ:

$$\|\varphi\| = C.$$

Let $v^0 \in l_\infty(\mathbb{Z}^s)$ be arbitrary, and fix $x \in \mathbb{R}^s$. Denote by $M = \operatorname{supp}\varphi \cap \mathbb{Z}^s$ and by $\Omega(x) = \{\alpha \in \mathbb{Z}^s : x \in \operatorname{supp}\varphi(\cdot - \alpha)\}$. M is finite, since φ has compact support, and $|\Omega(x)| \leq M + s$. Using (4.14), we obtain

$$|S^\infty v^0(x)| = |\sum_{\alpha \in \mathbb{Z}^s} v^0_\alpha \varphi(x - \alpha)| = |\sum_{\alpha \in \Omega(x)} v^0_\alpha \varphi(x - \alpha)| \leq$$
$$\leq C \sum_{\alpha \in \Omega(x)} |v^0_\alpha| \leq (M + s)C\|v^0\|.$$

Hence, $\|S^\infty v^0\| \leq (M + s)C\|v^0\|$, and S is stable. $\square$

4.2.1 The univariate case

In this subsection, we consider only the case $s = 1$! Let us start with defining the important notion of polynomial reproduction.

Definition 4.2 *Let S be a stationary subdivision scheme. S has* ***order of polynomial reproduction*** *$N \in \mathbb{N}$, if S reproduces constants, and N is the maximal number with the property: for every $n < N$, and every monic polynomial $p(x)$ of degree n, there exists a monic polynomial $q(x)$ of degree n such that*

$$Sp^0 = q^1.$$

We also say that S ***reproduces polynomials*** *of degree n, for every $n < N$.*

In other words, S reproduces polynomials of degree n, if $\Pi_n(\mathbb{R}) \setminus \Pi_{n-1}(\mathbb{R})$ stays invariant under the action of S. There is a natural question: does polynomial reproduction of degree n guarantee polynomial reproduction of degree $m \in \mathbb{N}$, for every $m < n$? The answer is affirmative, but only when in addition S reproduces constants. Different proofs can be found in Section 2 in [Han01] and in Proposition 2.4 in [JZ04].

Proposition 4.3 *Let S be a linear, shift invariant (not necessary finite) subdivision scheme that reproduces constants. Then there exists a linear subdivision scheme S_1, such that*

$$\triangle^1 \circ S = S_1 \circ \triangle^1. \tag{4.19}$$

S_1 is called the first derived scheme of S.

Proof. The main trick in this proof is to use symbolic calculus.
For $v^0 \in l_\infty(\mathbb{Z})$, the *generating function* $V_0(z)$ is given by the Laurent series

$$V_0(z) = \sum_{k\in\mathbb{Z}} v_k z^k.$$

Let $a(z)$ be the generating function of the mask $\mathbf{a}$ of S. We will refer to it as the *symbol* of the subdivision scheme and, obviously, S is completely determined by its symbol. Then

$$V_{j+1}(z) = \sum_{k\in\mathbb{Z}} v_k^{j+1} z^k = \sum_{k\in\mathbb{Z}} \Big(\sum_{l\in\mathbb{Z}} a_{k-rl} v_l^j \Big) z^k = \sum_{l\in\mathbb{Z}} v_l^j z^{rl} \sum_{k\in\mathbb{Z}} a_k z^k = a(z) V_j(z^r). \tag{4.20}$$

Here, r is the dilation factor of S.
Denote by

$$D_j(z) = \sum_{k\in\mathbb{Z}} (\triangle^1 v^j)_k z^k = \sum_{k\in\mathbb{Z}} (v_{k+1}^j - v_k^j) z^k = V_j(z)\frac{1-z}{z} \tag{4.21}$$

the generating function of $\triangle^1 S^j v^0$. Using (4.20) and (4.21), we get

$$\begin{aligned} D_{j+1}(z) &= V_{j+1}(z)\frac{1-z}{z} = a(z)V_j(z^r)\frac{1-z}{z} = \\ &= a(z)\frac{1-z}{z}\frac{z^r}{1-z^r} D_j(z) = a^{(1)}(z) D_j(z), \end{aligned} \tag{4.22}$$

with

$$a^{(1)}(z) = \frac{z^{r-1}a(z)}{z^{r-1} + z^{r-2} + \cdots + 1}. \tag{4.23}$$

Thus, to finish our prove it suffices to show that $a^{(1)}(z)$ is a generating function. Let ξ be a primitive r-th root of 1. We need to prove that $a(\xi) = 0$. Indeed,

$$a(\xi) = \sum_{i\in\mathbb{Z}} a_i \xi^i = \sum_{l=0}^{r-1} \Big(\sum_{j\in\mathbb{Z}} a_{l-rj}\Big)\xi^l = \sum_{l=0}^{r-1} \xi^l = 0 \tag{4.24}$$

For the third equality above we used (4.11) and that S reproduces constants. □

Remark 4.4 *Note that we not only proved existence of a first derived scheme, but also gave an algorithm for its construction. Moreover, if S is finite* (4.23) *gives*

$$|\mathrm{supp}\,(a^{(1)})| \leq |\mathrm{supp}\,(a)|. \tag{4.25}$$

By induction, if N is the order of polynomial reproduction of S, for each $n \leq N$ there exists an n-th derived scheme S_n such that

$$\triangle^n \circ S = S_n \circ \triangle^n.$$

Moreover, if the S-refinable function $\varphi \in C^l(\mathbb{R})$, we have

$$\frac{d^i}{dx^i} S^\infty v^0 = S_i^\infty \triangle^i v^0, \qquad v^0 \in l_\infty(\mathbb{Z}),\ \ \forall i : 1 \leq i \leq \max(N-1, l). \tag{4.26}$$

Definition 4.5 *Let $S : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ be a linear and bounded (with respect to the operator norm) operator. The* ***spectral radius*** *of S is*

$$\rho_\infty(S) = \liminf_{j\to\infty} \|S^j\|^{1/j}. \tag{4.27}$$

Theorem 4.6 *A linear, local subdivision scheme S is uniformly convergent, if and only if S reproduces constants and $\rho_\infty(S_1) < 1$.*

Proof. We follow the proof from [Dyn92].
Let S be uniformly convergent and $v^0 \in l_\infty(\mathbb{Z})$. Hence, S reproduces constants and S_1 exists. Then for any $k \in \mathbb{Z}$

$$\begin{aligned} &|(S_1^j \triangle^1 v^0)_k| = |(\triangle^1 v^j)_k| = |v^j_{k+1} - v^j_k| \leq \\ &\leq |v^j_{k+1} - S^\infty v^0(r^{-j}(k+1))| + |v^j_k - S^\infty v^0(r^{-j}k)| + \\ &+ |S^\infty v^0(r^{-j}(k+1)) - S^\infty v^0(r^{-j}k)| \xrightarrow[j\to\infty]{} 0 \end{aligned}$$

Since S converges uniformly, and the refinable function φ is compactly supported and thus, uniformly continuous, S_1 converges uniformly to the zero function for all initial data $v^0 \in l_\infty(\mathbb{Z})$. It

remains to show that the uniform convergence of S_1 to zero is equivalent to $\rho_\infty(S_1) < 1$.
Take $v^0 \in l_\infty(\mathbb{Z})$ with $\|v^0\| = 1$. For any $\epsilon > 0$, any $k \in \mathbb{Z}$, and $j > J'(\epsilon)$ we have

$$|(S_1^j v^0)_k| = \left|\Big(\sum_{l\in\mathbb{Z}} v_l^0 S_1^j \delta_l\Big)_k\right| = \left|\sum_{l\in\mathbb{Z}} v_l^0 (S_1^j\delta_l)_k\right| \leq \sum_{l\in\mathbb{Z}} |(S_1^j\delta)_{k-r^j l}| \leq M\|S_1^j\delta\| < \epsilon,$$

where $M = |\mathrm{supp}\,(\mathbf{a})|$, and $J'(\epsilon)$ comes from the uniform convergence to zero of S_1. We used (4.25) and that, due to (4.13) and induction, $\mathrm{supp}\,(S_1^j\delta) \subset (r^j - 1)\mathrm{supp}\,(a^{(1)})$.
Therefore, there exists a positive integer J, and $\mu \in (0,1)$, such that for all $v^0 \in l_\infty(\mathbb{Z})$

$$\|S_1^J v^0\| < \mu\|v^0\|, \tag{4.28}$$

which gives

$$\rho_\infty(S_1) < \mu < 1. \tag{4.29}$$

On the other hand, suppose that $\rho_\infty(S_1) < 1$. Let f^j be as in Definition 2.6. Than, we have to show that the sequence $\{f^j(x) : j \in \mathbb{Z}_+\}$ converges uniformly in $L_\infty(\mathbb{R})$, which, since $L_\infty(\mathbb{R})$ is a Banach space, is equivalent to showing that the sequence is Cauchy.

Observe that f^j is the limit curve of v^j for the scheme $S_{L,r}$ with dilation factor r and refinable function - the B-spline $B_1(x)$ of degree 1. For example, the scheme defined via (4.7) is exactly $S_{L,2}$. $S_{L,r}$ reproduces constants due to (4.1) and (4.15). Oswald has the following result:

Lemma 4.7 *Let $T : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ be a bounded linear operator with local support, such that $T\mathbf{1} = \mathbf{0}$. Then there exists, a bounded linear operator $\hat{T}$ with the same support as T which satisfies*

$$Tv = \hat{T}\triangle v, \quad \forall v \in l_\infty(\mathbb{Z}).$$

The proof can be found in [Osw02].
Now, the linear operator $T = S - S_{L,r}$ has local support and $T\mathbf{1} = \mathbf{0}$, since both S and $S_{L,r}$ are local and reproduce constants. Moreover, T is shift-invariant and thus - bounded. By Lemma 4.7

$$\|(S - S_{L,r})v^j\| \leq E\|\triangle^1 v^j\| \leq 2E\|v^j\|, \tag{4.30}$$

with some positive constant $E < \infty$.

Using (4.17) and that $B_1(x)$ is the $S_{L,r}$-refinable function, we derive

$$f^{j+1}(x) - f^j(x) = \sum_{k\in\mathbb{Z}}((S - S_{L,r})v^j)_k B_1(r^{j+1}x - k). \tag{4.31}$$

Now, combining the last two results with (4.1), the fact that B_1 is non-negative, and (4.28), we derive

$$|f^{j+1}(x) - f^j(x)| \leq \|(S - S_{L,r})v^j\| \leq 2E\|v^j\| \leq 2E \max_{0\leq i<J}\|v^i\|\mu^{[j/J]}. \tag{4.32}$$

The geometrical decay in (4.32) assures that $\{f^j(x) : j \in \mathbb{Z}_+\}$ is a Cauchy sequence and completes the proof. $\square$

Theorem 4.6 and Remark 4.4 give an explicit algorithm for checking uniform convergence of a local, linear subdivision scheme. In order to prove uniform convergence, we need to find the first $J \in \mathbb{Z}$ such that $\|S_1^J\| < 1$. From theoretical point of view the number of iterations J is not uniformly bounded (for every $J \in \mathbb{N}$ one can always find a uniformly convergent scheme S for which $\|S_1^j\| \geq 1, \forall j \leq J$), but from practical point of view, *if* $J > 10$ no convergence occurs in the actual performance of the scheme, since only a small number of steps ($j < 10$) are carried out in practice.

4.2.2 Examples

We denote by $S_{mid,r}$ the midpoint-interpolating subdivision scheme with dilation factor r, where $2 \le r \in \mathbb{N}$. More precisely, for any $v^0 \in l_\infty(\mathbb{Z})$ and any $k \in \mathbb{Z}$ take the unique polynomial $p_k \in \Pi_2(\mathbb{R})$ that interpolates the data $\{v_{k-1}, v_k, v_{k+1}\}$, i.e.

$$p_k(k+l) = v^0_{k+l}, \qquad l \in \big\{-1, 0, 1\big\}. \tag{4.33}$$

Then

$$(S_{mid,r}v^0)_{rk+l} = p_k\Big(k + \frac{2l+1-2r}{2r}\Big), \qquad l \in \big\{1-[r], \dots, [r]\big\}. \tag{4.34}$$

The last equation means, that we have divided the interval $[k-1/2, k+1/2]$ into r equally long subintervals and use the values of $p_k(x)$ in the midpoints for constructing v^1. Using the Lagrange formula for interpolation, we derive

$$p_0(x) = \frac{x(x-1)}{2}v^0_{-1} + (1-x^2)v^0_0 + \frac{x(x+1)}{2}v^0_1. \tag{4.35}$$

Obviously, $S_{mid,r}$ is linear, shift-invariant and local subdivision scheme with order of polynomial reproduction 3. (4.33) is usually known as the *interpolation* step, and (4.34) - as the *imputation* step in defining subdivision. Most of the (not necessarily linear) schemes used in practise follow these two steps: (W)ENO (see [CDM03]), PPH (see [AL05]), median-interpolating subdivision scheme (see [DY00]), etc.
Amoung the family of subdivision schemes $\{S_{mid,r} : 2 \le r \in \mathbb{N}\}$, $S_{mid,2}$ and $S_{mid,3}$ have greatest importance in applications.

Example 4.8 $S = S_{mid,2}$. *Using*(4.33), (4.34), *and* (4.35) *we get:*

$$(S_{mid,2}v^0)_{2k} = \frac{5}{32}v^0_{k-1} + \frac{15}{16}v^0_k - \frac{3}{32}v^0_{k+1}, \qquad (S_{mid,2}v^0)_{2k+1} = -\frac{3}{32}v^0_{k-1} + \frac{15}{16}v^0_k + \frac{5}{32}v^0_{k+1}. \tag{4.36}$$

The symbol of $S_{mid,2}$ *is*

$$a(z) = -\frac{3}{32}z^{-2} + \frac{5}{32}z^{-1} + \frac{15}{16} + \frac{15}{16}z + \frac{5}{32}z^2 - \frac{3}{32}z^3.$$

Therefore

$$a^{(1)}(z) = \frac{z}{z+1}a(z) = -\frac{3}{32}z^{-1} + \frac{1}{4} + \frac{11}{16}z + \frac{1}{4}z^2 - \frac{3}{32}z^3,$$

and

$$(S_1v^0)_{2k} = \frac{1}{4}v^0_{k-1} + \frac{1}{4}v^0_k, \qquad (S_1v^0)_{2k+1} = -\frac{3}{32}v^0_{k-1} + \frac{11}{16}v^0_k - \frac{3}{32}v^0_{k+1}.$$

Since $\rho_\infty(S_1) \le \|S_1\| = \frac{15}{16} < 1$, $S_{mid,2}$ *is uniformly convergent.*
For the higher-ordered derived schemes, we get

$$a^{(2)}(z) = \frac{z}{z+1}a^{(1)}(z) = -\frac{3}{32} + \frac{11}{32}z + \frac{11}{32}z^2 - \frac{3}{32}z^3, \qquad \|S_2\| = \frac{7}{16};$$
$$a^{(3)}(z) = \frac{z}{z+1}a^{(2)}(z) = -\frac{3}{32}z + \frac{7}{16}z^2 - \frac{3}{32}z^3, \qquad \|S_3\| = \frac{7}{16}.$$

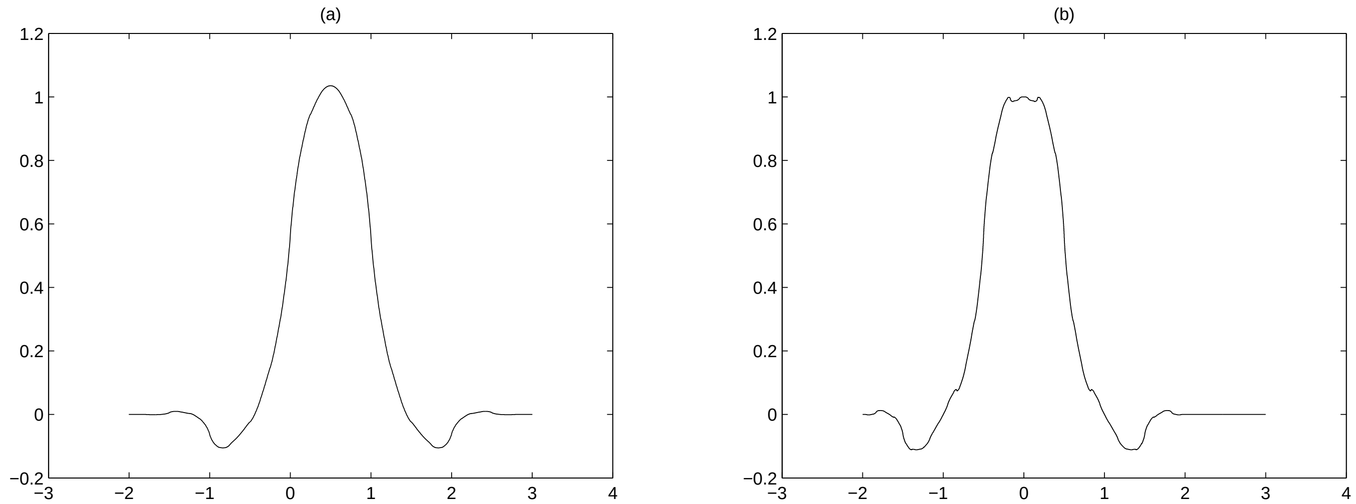

Figure 5: The refinable functions for: (a) $S_{mid,2}$, (b) $S_{mid,3}$.

Example 4.9 $S = S_{mid,3}$. *Analogously to Example 4.8*

$$(Sv^0)_{3k-1} = \frac{2}{9}v^0_{k-1} + \frac{8}{9}v^0_k - \frac{1}{9}v^0_{k+1}, \quad (Sv^0)_{3k} = v^0_k, \quad (Sv^0)_{3k+1} = -\frac{1}{9}v^0_{k-1} + \frac{8}{9}v^0_k + \frac{2}{9}v^0_{k+1}. \quad (4.37)$$

$$a(z) = -\frac{1}{9}z^{-4} + \frac{2}{9}z^{-2} + \frac{8}{9}z^{-1} + 1 + \frac{8}{9}z + \frac{2}{9}z^2 - \frac{1}{9}z^4;$$
$$a^{(1)}(z) = \frac{z^2}{z^2+z+1}a(z) = -\frac{1}{9}z^{-2} + \frac{1}{9}z^{-1} + \frac{2}{9} + \frac{5}{9}z + \frac{2}{9}z^2 + \frac{1}{9}z^3 - \frac{1}{9}z^4;$$
$$(S_1v^0)_{3k-1} = \frac{2}{9}v^0_{k-1} + \frac{1}{9}v^0_k, \quad (Sv^0)_{3k} = \frac{1}{9}v^0_{k-1} + \frac{2}{9}v^0_k, \quad (Sv^0)_{3k+1} = -\frac{1}{9}v^0_{k-1} + \frac{5}{9}v^0_k - \frac{1}{9}v^0_{k+1};$$
$$\|S_1\| = \frac{7}{9} < 1.$$

Therefore, $S_{mid,3}$ *is uniformly convergent.*
For the higher-ordered derived schemes, we get

$$a^{(2)}(z) = \frac{z^2}{z^2+z+1}a^{(1)}(z) = -\frac{1}{9} + \frac{2}{9}z + \frac{1}{9}z^2 + \frac{2}{9}z^3 - \frac{1}{9}z^4, \qquad \|S_2\| = \frac{1}{3};$$
$$a^{(3)}(z) = \frac{z^2}{z^2+z+1}a^{(2)}(z) = -\frac{1}{9}z^2 + \frac{1}{3}z^3 - \frac{1}{9}z^4, \qquad \|S_2\| = \frac{1}{3}.$$

4.2.3 The multivariate case

After establishing systematic theory for stability of local, linear, univariate subdivision schemes in Section 4.2.1, the main question is how far this theory can be extended. Two immediate candidates are the nonlinear univariate subdivision schemes and the linear multivariate ones. In this section we will show that for the second type of schemes there is a generalization of Theorem 4.6, which is very natural but, unfortunately, not so applicable in practice.

Let $e = (1, \ldots, 1) \in \mathbb{Z}^s$, and $e^{(i)} : e^{(i)}_j = \delta_{ij}, j = 1, \ldots, s$ be the i-th coordinate vector in $\mathbb{Z}^s$. Then we can extend the notion of a first derived difference in higher dimensions:

$$\triangle^1 : l_\infty(\mathbb{Z}^s) \to (l_\infty(\mathbb{Z}^s))^s, \qquad (\triangle^1 v)_\alpha = \left\{ v_{\alpha+e^{(i)}} - v_\alpha : 1 \le i \le s \right\}.$$

A detailed analysis of the proof of Theorem 4.6 gives that it was based on the existence of a derived scheme S_1, Lemma 4.7 and (4.28).

In $s > 1$ uniform convergence again implies reproduction of constants, and, analogously to the case $s = 1$, reproduction of constants implies existence of a local, linear subdivision scheme $S_1 : (l_\infty(\mathbb{Z}^s))^s \to (l_\infty(\mathbb{Z}^s))^s$ (see Proposition 6.8. in [Dyn92]), such that

$$\triangle^1 \circ S = S_1 \circ \triangle^1.$$

It is not difficult to prove the multivariate version of Lemma 4.7, either (see Proposition 6.7. in [Dyn92]). We only have to be careful with the fact that, unlike in $1D$, $\triangle^1$ is not an endomorphism and the image space $(l_\infty(\mathbb{Z}^s))^s$ is equipped with the infinite-norm

$$\|\triangle^1 v\| = \sup_{\alpha \in \mathbb{Z}^s} \|\triangle^1 v_\alpha\|,$$

where $\|\triangle^1 v_\alpha\| = \max_{1 \le i \le s} |(\triangle^1 v_\alpha)_i|$.

Now, we are ready the extend the stability analysis to the multivariate case.

Theorem 4.10 *Let $S : l_\infty(\mathbb{Z}^s) \to l_\infty(\mathbb{Z}^s)$ be local, linear subdivision scheme that reproduces constants. Then S is uniformly convergent, if and only if there exist $J \in \mathbb{Z}_+$ and $\tilde{\mu} \in (0,1)$ such that*

$$\|S_1^J \triangle^1 v^0\| \le \tilde{\mu} \|\triangle^1 v^0\|, \qquad \forall v^0 \in l_\infty(\mathbb{Z}^s). \tag{4.38}$$

The proof follows that of Theorem 4.6.

Although Theorem 4.10 is nothing but "translating" all the definitions from the one-dimensional case, to the "language" of the multidimensional one, its application in practice is tremendously smaller. The problem is again in $\triangle^1$ and that, unlike the univariate case, this linear operator is not surjective! For example, if $v \in l_\infty(\mathbb{Z}^s)$, then $\triangle^1 v$ satisfies relations like

$$e^{(j)} \cdot [(\triangle^1 v)_{\alpha+e^{(i)}} - (\triangle^1 v)_\alpha] = e^{(i)} \cdot [(\triangle^1 v)_{\alpha+e^{(j)}} - (\triangle^1 v)_\alpha], \qquad \alpha \in \mathbb{Z}^s,$$

for every $i, j \in \{1, \ldots, s\}$.
This means, that we don't have formulation of Theorem 4.10 involving the spectral radius of the subdivision scheme S_1 and thus, no lower bound for $\tilde{\mu}$. (compare with (4.29)!) In other words, to check whether S_1 satisfies (4.38) is sometimes very complicated, which makes (4.38) "bad sufficient condition" for applications.

The latter can be improved by a one-level higher generalization on (4.28).

Definition 4.11 *Let $D : l_\infty(\mathbb{Z}^s) \to [0,\infty)$ be a non-trivial function. We say that S is* ***contractive relative*** *to D, if there exists $\mu \in (0,1)$ and $J \in \mathbb{Z}_+$ such that for all $v \in l_\infty(\mathbb{Z}^s)$*

$$D(S^J v) \leq \mu D(v). \tag{4.39}$$

For example, both (4.28) and (4.38) mean that S_1 is contractive relative to $D(v) = \|\triangle^1 v\|$. We will see later, that this extension happens to play a role in the stability analysis on a univariate, nonlinear subdivision as well.

Theorem 4.12 *Let S be local, linear subdivision scheme that reproduces constants and is contractive relative to D. Let $\tilde{S}$ be a uniformly convergent subdivision scheme with a mask $\tilde{\mathbf{a}}$ and $\tilde{S}$-refinable function, $\varphi \in C(\mathbb{R}^s)$, satisfying the "stability hypothesis"*

$$c_1\|v\| \leq \left\| \sum_{\alpha\in\mathbb{Z}^s} v_\alpha \varphi(x-\alpha) \right\| \leq c_2\|v\|, \qquad v \in l_\infty(\mathbb{Z}^s). \tag{4.40}$$

If

$$\|(S-\tilde{S})v\| \leq cD(v), \qquad v \in l_\infty(\mathbb{Z}^s), \tag{4.41}$$

then S converges uniformly.

The proof follows directly from that of Theorem 4.6 with φ instead of B_1.

Not fixing D a priori, makes the theorem more flexible and applicable. The only question is: do we know enough appropriate uniformly convergent linear multivariate subdivision schemes? Dyn in [Dyn92] shows that a tensor product of s copies of a uniformly convergent univariate linear subdivision scheme is uniformly convergent (Proposition. 6.3.), as well as local, linear subdivision schemes with positive mask (Theorem 6.4.).

5 Univariate nonlinear subdivision. Examples

5.1 Cubic ENO and WENO

The *essentially non-oscillatory* (ENO) and the *weighted* essentially non-oscillatory (WENO) subdivision schemes are interpolatory and dyadic, i.e.

$$(Sv^j)_{2k} = v^j_k \qquad \forall j,k \in \mathbb{Z}.$$

Thus, in order to determine these schemes, we only need to give a formula for the odd entries of Sv^j.

Let us fix a *cost function* $\varphi : \Pi_3(\mathbb{R}) \to [0,\infty)$. Let $\Gamma^j = 2^{-j}\mathbb{Z}$, $k \in \mathbb{Z}$, and $v^j \in l_\infty(\Gamma^j)$.

- *(the interpolation step)* Compute the cubic polynomials $p^j_{k,i}$, $i = 0,1,2$ that interpolate

$$\left\{\left(2^{-j}(k-2+i+l); v_{k-2+i+l}\right)\right\}, l = 0,\ldots,3.$$

- *(the imputation step)* Among these polynomials take the least oscillatory one, i.e. the one that minimizes φ. Let us denote it by p_k^j. Then

$$(S_{ENO}v^j)_{2k+1} = p_k^j(2^{-(j+1)}(2k+1)).$$

The ENO scheme was originally proposed by Harten, Engquist, Osher and Chakravarthy in [HEOC87]. It is one of the most robust algorithms for avoiding oscillations of a continuous approximation of data, near points of discontinuity (the so called *Gibbs phenomena*). A simple example is given in fig. 6.

The set of grid points, used for constructing p_k^j is called *stencil*. The interpolation step gives that for a fixed $k \in \mathbb{Z}$ there are three candidates for stencil at k, namely

$$\left\{2^{-j}(k-1-l), 2^{-j}(k-l), 2^{-j}(k+1-l), 2^{-j}(k+2-l)\right\}, \qquad l = -1, 0, 1,$$

to which we will refer as *right*, *central*, and *left* stencils. It is obvious, that ENO has order of polynomial reproduction 4 and, although defined in terms of the grid Γ^j, doesn't depend on the level of subdivision, i.e. is stationary. If for some k the least oscillatory polynomial is not unique, there is no prescribed rule about which one should be taken in the imputation step.

Although globally nonlinear, ENO is locally linear, i.e. for every k, $(S_{ENO}v)_{2k+1}$ is a linear combination of the values of v on its stencil. More precisely, for $v \in l_\infty(\mathbb{Z})$ let

$$(S_l v)_{2k} = v_k \qquad (S_l v)_{2k+1} = \frac{1}{16}v_{k-2} - \frac{5}{16}v_{k-1} + \frac{15}{16}v_k + \frac{5}{16}v_{k+1} \tag{5.1}$$

$$(S_c v)_{2k} = v_k \qquad (S_c v)_{2k+1} = -\frac{1}{16}v_{k-1} + \frac{9}{16}v_k + \frac{9}{16}v_{k+1} - \frac{1}{16}v_{k+2} \tag{5.2}$$

$$(S_r v)_{2k} = v_k \qquad (S_r v)_{2k+1} = \frac{5}{16}v_k + \frac{15}{16}v_{k+1} - \frac{5}{16}v_{k+2} + \frac{1}{16}v_{k+3} \tag{5.3}$$

be the linear subdivision schemes, that at every $k \in \mathbb{Z}$ take: (5.1) the left, (5.2) the central, or (5.3) the right stencil. Then we can represent ENO by a map $\Phi : l_\infty(\mathbb{Z}) \to \mathfrak{L} := \{S_l, S_c, S_r\}$, such that

$$S_{ENO}v = \Phi(v)v. \tag{5.4}$$

Remark 5.1 *The above representation holds in general, i.e. for any nonlinear subdivision scheme S, one can find a set $\mathfrak{L}$ of linear subdivision rules with the same (or smaller) support as S, such that* (5.4) *is satisfied. Φ is sometimes called* quasi-linear *map (see [CDM03]). The minimal order of polynomial reproduction of the elements of $\mathfrak{L}$ is called order of polynomial reproduction of Φ. From* (5.4) *it follows that the order of polynomial reproduction of Φ is less or equal to that of S. For a large class of subdivision schemes the two orders coincide (ENO, WENO, etc.), but there are some schemes (PPH), for which the "quasi-linear order" is strictly smaller than the subdivision one. Since for every $v \in l_\infty(\mathbb{Z})$ the action of $\Phi(v)$ is known a priori only on v, Φ is by no means unique! The questions about finding the set $\mathfrak{L}$ with minimal cardinality, and the quasi-linear map Φ with maximal order of reproduction depend heavily on S and are still open.*

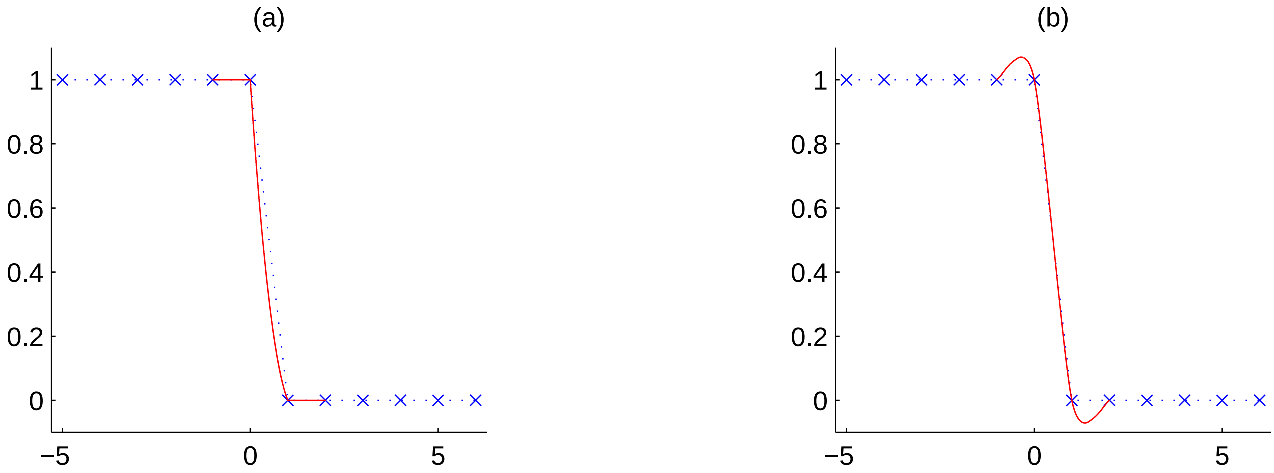

Figure 6: The action of: (a) ENO, (b) S_c, on the step function.

Let S_l^1, S_c^1, and S_r^1 be the corresponding first derived schemes. From proposition 4.3 and (5.4) it follows that there exists an operator-valued function $\tilde{\Phi} : l_\infty(\mathbb{Z}) \to \mathfrak{L}' := \{S_l^1, S_c^1, S_r^1\}$, such that

$$\triangle^1(S_{ENO}v) = \tilde{\Phi}(v)\triangle^1 v, \qquad v \in l_\infty(\mathbb{Z}). \tag{5.5}$$

In [CDM03] it is proved that ENO converges for any cost function φ. On the other hand since it uses only one of the candidate stencils and small perturbation of the initial data could change the stencil, ENO is unstable. This problem can be solved by a slight modification of the ENO interpolation technique - WENO.

WENO uses the same idea as ENO, but instead of fixing the stencil, one works with a convex combination of the interpolating polynomials $\{p_{k,i}^j\}_{i=1}^3$, i.e. the polynomial p_k^j, used in the imputation step is of the form

$$p_k^j = \alpha_{k,1}p_{k,1}^j + \alpha_{k,2}p_{k,2}^j + \alpha_{k,3}p_{k,3}^j, \qquad \alpha_{k,1} + \alpha_{k,2} + \alpha_{k,3} = 1, \tag{5.6}$$

with nonnegative weights that depend on the initial data and the position k, but not on the level of subdivision.

This scheme was originally proposed by Liu, Osher, and Chan in [LOC94]. There, they used

$$\varphi(p_{k,i}^j) = 2^{-j}\int_{2^{-j}k}^{2^{-j}(k+1)} \left((p_{k,i}^j)'\right)^2 dx + 2^{-3j}\int_{2^{-j}k}^{2^{-j}(k+1)} \left((p_{k,i}^j)''\right)^2 dx \tag{5.7}$$

as cost function, and

$$\alpha_{k,i} = \frac{a_{k,i}}{a_{k,1} + a_{k,2} + a_{k,3}}, \qquad i = 1, 2, 3,$$

where

$$a_{k,i} := \frac{d_i}{(\epsilon + b_{k,i})^2}, \qquad b_{k,i} = \varphi(p_{k,i}^j), \tag{5.8}$$

with positive constants $\{d_i\}_{i=1}^3$ and ϵ.

Since the interpolating polynomials depend on the grid Γ^j, to make WENO a stationary subdivision scheme, one should use normalization factors in the definition of the cost function (see (5.7)). (5.8) implies that, in order WENO to be shift-invariant, $\varphi(p^j_{k,i})$ must be a function of the first derivative of $p^j_{k,i}$ rather than of the polynomial itself. Thus, the constant functions lie in the kernel of φ and the additional parameter ϵ in (5.8) is necessary for defining $a_{k,i}$ correctly. On the other hand, introducing ϵ makes WENO not affine-invariant: the scheme is still translation invariant, but direct computation gives

$$p^j_{k,i}(\lambda v) = \lambda p^j_{k,i}(v)$$

and

$$\alpha_{k,i}(\lambda v) = \frac{\frac{d_i}{(\lambda^{-2}\epsilon + b_{k,i}(v))^2}}{\frac{d_1}{(\lambda^{-2}\epsilon + b_{k,1}(v))^2} + \frac{d_2}{(\lambda^{-2}\epsilon + b_{k,2}(v))^2} + \frac{d_3}{(\lambda^{-2}\epsilon + b_{k,3}(v))^2}} \neq \alpha_{k,i}(v),$$

where $\lambda \in \mathbb{R}$! Therefore, the limiting behavior of WENO when ϵ tends to zero is an important question, but, up to the knowledge of the author, still without an answer.
The $b_{k,i}$'s are called "smoothness indicators" and they preserve the non-oscillatory property of the weighted ENO, i.e. $\alpha_{k,i}$ is "big" when $\varphi(p^j_{k,i})$ is "small" and vice versa. Note that, due to the previous remarks, $b_{k,i}$ and hence, $\alpha_{k,i}$ are functions on $\triangle^1 v$ rather than on v!
The positive constants $\{d_i\}_{i=1}^3$ are, in general, arbitrary. The order of polynomial reproduction of WENO is at least the one of ENO - 4, but it can be increased by one using a special choice of the d_i's (see [LOC94] for details).

We can modify (5.4) and derive the corresponding representation for WENO. We just need to take $\mathfrak{L}$ to be the "convex hull" of $\{S_l, S_c, S_r\}$, i.e.

$$\Phi(v) = \alpha_l S_l + \alpha_c S_c + \alpha_r S_r, \qquad v \in l_\infty(\mathbb{Z}), \tag{5.9}$$

where $\alpha_l : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ is the nonlinear operator with respect to $\alpha_{\cdot,1}$: $(\alpha_l v)_{2k+1} := \alpha_{k,1}(v)$, $(\alpha_l v)_{2k} := 1/3$, and $\alpha_l S_l$ is their "inner product" $\big((\alpha_l S_l)v\big)_k := (\alpha_l v)_k (S_l v)_k$. (Analogously for α_c, and α_r). Moreover, due to the special type of the cost function, there exists a $\tilde{\Phi} : l_\infty(\mathbb{Z}) \to \mathfrak{L}$, such that $\Phi(v) = \tilde{\Phi}(\triangle^1 v)$:

$$S_{WENO} v = \Phi(v)v = \tilde{\Phi}(\triangle^1 v)v. \tag{5.10}$$

Cohen, Dyn, and Matei proved in [CDM03] that, unlike ENO, the weighted essentially non-oscillatory scheme is stable (in a weakened version of stability)! The main drawback of this proof is that it heavily depends on ϵ and does not provide us with any information about the limit case $\epsilon \to 0$.

5.2 PPH

The *piecewise polynomial harmonic* (PPH) subdivision scheme is again interpolatory and dyadic, with

$$(S_{PPH} v)_{2k+1} = \frac{v_k + v_{k+1}}{2} - \frac{1}{8} H(\triangle^2 v_{k-1}, \triangle^2 v_k), \tag{5.11}$$

where $H:\mathbb{R}^2\to\mathbb{R}$ is the harmonic mean defined by

$$H(x,y)=\begin{cases}\frac{2xy}{x+y}, & xy>0\\ 0, & otherwise\end{cases}.$$

(5.11) is the explicit result of the following algorithm:

Let $\Gamma^j=2^{-j}\mathbb{Z}$, $k\in\mathbb{Z}$, and $v^j\in l_\infty(\mathbb{Z})$.

- *(the interpolation step)* Compute the cubic polynomial p_k^j that interpolates the data

x	$2^{-j}(k-1)$	$2^{-j}k$	$2^{-j}(k+1)$	$2^{-j}(k+2)$
$p_k^j(x)$	v_{k-1}^j	v_k^j	v_{k+1}^j	$v_{k+1}^j+v_k^j-v_{k-1}^j+2H(\triangle^2 v_{k-1},\triangle^2 v_k)$

- *(the imputation step)*

$$(S_{PPH}v)_{2k+1}=p_k^j(2^{-j-1}(2k+1)) \tag{5.12}$$

The PPH subdivision scheme was introduced by Kuijt and van Damme in [KvD98a]. PPH is stationary and affine invariant, due to (5.11), and has order of polynomial reproduction 3, due to (5.12), the fact that $\triangle_h^2 p(x)=2h^2a_0$ for any $p\in\Pi_2(\mathbb{R})$ with $p(x)=a_0x^2+\ldots$ and any $x,h\in\mathbb{R}$, the trivial equality

$$H(x,x)=A(x,x),\qquad \forall x\in\mathbb{R},$$

where

$$A(x,y)=\frac{x+y}{2}$$

is the arithmetic mean, and the observation

$$v_{k+2}^j=v_{k+1}^j+v_k^j-v_{k-1}^j+2A(\triangle^2 v_{k-1},\triangle^2 v_k). \tag{5.13}$$

Let us prove that every quasi-linear operator-valued function Φ with respect to PPH has strictly lower order of polynomial reproduction than PPH. The general form of Φ is

$$(\Phi(v)w)_{2k}=w_k;\qquad (\Phi(v)w)_{2k+1}=\sum_{l=k-1}^{k+2}a_{2k+1,l}(v)w_l.$$

In order $\Phi(v)$ to reproduce constants, linear and quadratic polynomials, as well as to satisfy $S_{PPH}v=\Phi(v)v$, the following 4x4 linear system for the unknown coefficients $\{a_{2k+1,l}\}_{l=k-1}^{k+2}$ should be satisfied:

$$\left|\begin{array}{l} a_{2k+1,k-1}+a_{2k+1,k}+a_{2k+1,k+1}+a_{2k+1,k+2}=1\\ -a_{2k+1,k-1}\qquad +a_{2k+1,k+1}+2a_{2k+1,k+2}=1/2\\ a_{2k+1,k-1}\qquad +a_{2k+1,k+1}+4a_{2k+1,k+2}=1/4\\ v_{k-1}a_{2k+1,k-1}+v_ka_{2k+1,k}+v_{k+1}a_{2k+1,k+1}+v_{k+2}a_{2k+1,k+2}=\frac{v_k+v_{k+1}}{2}-\frac{1}{8}H(\triangle^2v_{k-1},\triangle^2v_k)\end{array}\right.$$

Using Gaussian elimination, we derive

$$(2\triangle^1 v_{k-1}+5\triangle^1 v_k-\triangle^1 v_{k+1})a_{2k+1,k+2}=\frac{\triangle^1 v_{k-1}+2\triangle^1 v_k+H(\triangle^2 v_{k-1},\triangle^2 v_k)}{4},$$

which doesn't have solution for $v_{k-1}=0; v_k=-1; v_{k+1}=0; v_{k+2}=3$.

There are plenty of quasi-linear maps Φ corresponding to PPH with order of polynomial reproduction 2. Among them we would like to mention

$$(\Phi(v)w)_{2k+1}=-C_k(v)w_{k-1}+\left(\frac{1}{2}+2C_k(v)\right)w_k+\left(\frac{1}{2}-C_k(v)\right)w_{k+1}, \tag{5.14}$$

where

$$C_k(v)=\frac{\mathrm{sgn}(\triangle^2 v_{k-1}\triangle^2 v_k)+1}{8}\frac{\triangle^2 v_k}{\triangle^2 v_{k-1}+\triangle^2 v_k}.$$

Here, $\mathrm{sgn}(x)=1$ if $x>0$, and $\mathrm{sgn}(x)=-1$, otherwise.
The function, defined in (5.14) has two important properties. First, the support of $\Phi(v)$ is strictly smaller then the support of PPH, and moreover this is the minimal support for a linear operator to reproduce quadratic polynomials. Second, C_k and thus Φ itself are functions not on v but on $\triangle^2 v$ which stresses the fact that PPH is a second order perturbation of the degree 1 centered Lagrange interpolation S_L (see (5.11)). Hence, we have

$$S_{PPH}v=\Phi(v)v=\Phi_1(\triangle^1 v)v=\Phi_2(\triangle^2 v)v, \tag{5.15}$$

where the functions Φ_1 and Φ_2 can be explicitly computed from Φ.

The main property of PPH is that it is convexity preserving, i.e., if the initial data is convex (concave), then it remains convex (concave) at any level of subdivision, and in particular, the limit function is also convex (concave).

Indeed, for any $k\in\mathbb{Z}$ direct computations give

$$\begin{aligned}
\triangle^2 v^{j+1}_{2k-2} &= \frac{1}{4}H(\triangle^2 v^j_{k-2},\triangle^2 v^j_{k-1});\\
\triangle^2 v^{j+1}_{2k-1} &= \frac{(\triangle^2 v^j_{k-1})^2}{2}\Big(\frac{1}{\triangle^2 v^j_{k-2}+\triangle^2 v^j_{k-1}}+\frac{1}{\triangle^2 v^j_{k-1}+\triangle^2 v^j_k}\Big);\\
\triangle^2 v^{j+1}_{2k} &= \frac{1}{4}H(\triangle^2 v^j_{k-1},\triangle^2 v^j_k).
\end{aligned} \tag{5.16}$$

Hence, if $\{v^j_{k+i}\}_{i=-2}^2$ is convex, i.e., $(\triangle^2 v^j)_{k-l}>0, l=0,1,2$, then so is $\{v^{j+1}_{2k+i}\}_{i=-2}^2$. (If the data is concave, take $\tilde{v}^j=-v^j$.)

Furthermore, (5.16) shows that if the initial data $\{v^0_{k-1},v^0_k,v^0_{k+1},v^0_{k+2}\}$ is neither convex nor concave, than on the interval $[k,k+1]$ the PPH subdivision coincides with S_L, hence it is purely linear. On the contrary, if the initial data $\{v^0_{k-1},v^0_k,v^0_{k+1},v^0_{k+2}\}$ is either convex or concave, the PPH subdivision on the interval $[k,k+1]$ is always nonlinear (see fig. 2). This "separation" of linearity and nonlinearity, along with the fact that S_L is convergent and stable due to Section 4.2.1, allows us to work only with convex data, in order to prove convergence and stability of PPH.

The piecewise polynomial harmonic subdivision scheme is convergent and stable. The latter was proved by Amat and Liandrat in [AL05] and their proof was the starting point of our research.

5.3 Quadratic, dyadic median-interpolating subdivision scheme

Let f be real-valued continuous function on a bounded interval I with positive Lebesgue measure ($m(I) > 0$). Then the *median* of f on I is defined by

$$med(f;I) := \sup\left\{\alpha : m(\{x : f(x) < \alpha\}) \leq \frac{1}{2}m(I)\right\}. \tag{5.17}$$

The quadratic, dyadic median-interpolating subdivision scheme is given by the following interpolation-imputation algorithm:
Let $\Gamma^j = 2^{-j}\mathbb{Z}$, $k \in \mathbb{Z}$, and $v^j \in l_\infty(\mathbb{Z})$.

- *(the interpolation step)* For the interval $I_k^j = [2^{-j}k, 2^{-j}(k+1)]$ find the quadratic polynomial p_k^j such that
$$med(p_k^j; I_{k+l}^j) = v_{k+l}^j, \qquad l = -1, 0, 1. \tag{5.18}$$
- *(the imputation step)*
$$(S_{med}v^j)_{2k+l} = med(p_k^j; I_{2k+l}^{j+1}), \qquad l = 0, 1. \tag{5.19}$$

This scheme is a member of a family of subdivision schemes introduced by Donoho and Yu in [DY00]. In our paper we chose to work exactly with this median-interpolating subdivision scheme, because there is an explicit formula for the action of the scheme, and there are less parameters in the dyadic case than in the r-adic one, which simplifies the computations. From now on we will skip the words "quadratic" and "dyadic" and will refer to (5.19) as median-interpolating subdivision scheme.

The median-interpolating scheme is stationary, finite, and shift-invariant with order of polynomial reproduction 3. It is affine invariant, as well, due to

$$med(af + b; I) = a.med(f; I) + b. \tag{5.20}$$

For any data $v \in l_\infty(\mathbb{Z})$ there exists a unique polynomial p_k that satisfies (5.18). Unfortunately the normal form of the polynomial $a_0x^2 + a_1x + a_2$ is messy and bad for applications. Therefore (see [Osw04]), we prefer to write p_k in the form

$$p_k(x) = a(x - c)^2 + b. \tag{5.21}$$

Note that all quadratic polynomials but the linear ones can be written in such a way. The linear polynomials are monotonic and for monotone functions the median coincides with the value of the function in the midpoint of the interval. Thus, in this case the median-interpolating scheme coincides with the midpoint-interpolating scheme $S_{mid,2}$ defined via (4.36) on $\mathbb{Z} + 1/2$ instead of $\mathbb{Z}$. We have already proven stability on the latter in Section. 4.2.2, therefore the linear polynomials doesn't play any role in the stability analysis on the median-interpolating subdivision scheme and without loss of generality, we can assume that the interpolating polynomials $p_k(x)$ are of the form (5.21).

Since S_{med} is shift-invariant, we can derive a formula only for p_0, which we will denote by p_{med}. In order to use symmetry and to simplify the computations and the final formulas, we will work not on the grid $\mathbb{Z}$ but on $2\mathbb{Z}-1$, i.e.,

$$v_k = med(p_{med}; [2k-1, 2k+1]), \qquad k = -1, 0, 1.$$

We will denote by $\delta := \triangle^2 v_{-1}$, by $d_k := \triangle^1 v_k, k = -1, 0$, and by p_{mid} the corresponding midpoint-interpolating polynomial (see (4.35)):

$$p_{mid}(x) = \frac{x(x-2)}{8} v_{-1} + \frac{4-x^2}{4} v_0 + \frac{x(x+2)}{8} v_1.$$

It is a trivial exercise to show that for any $p = a(x-c)^2 + b$ and any bounded interval $I = [\alpha, \beta]$

$$med(p; I) = \begin{cases} p(\frac{\alpha+\beta}{2}), & \left|c - \frac{\alpha+\beta}{2}\right| \geq \frac{|I|}{4}, \\ p\left(c \pm \frac{|I|}{4}\right) = p(\frac{\alpha+\beta}{2}) + \frac{a}{4}\left(\frac{|I|^2}{4} - 4(\frac{\alpha+\beta}{2} - c)^2\right), & \left|c - \frac{\alpha+\beta}{2}\right| < \frac{|I|}{4}. \end{cases} \tag{5.22}$$

In particular,

$$v_k = p_{med}(2k) + \frac{a}{4}\epsilon_{2k} = a(2k-c)^2 + b + \frac{a}{4}\epsilon_{2k}, \qquad k = -1, 0, 1, \tag{5.23}$$

where $\epsilon_n := (1-4(n-c)^2)_+ \in [0,1]$ depend on the center c of p_{mid} and vanish for $c \notin (n-1/2, n+1/2)$. Therefore, using that $\triangle_2^2 p_{med}(-2) = 8a$, we obtain

$$a = \frac{4\delta}{32 + \epsilon_{-2} - 2\epsilon_0 + \epsilon_2}. \tag{5.24}$$

Equations (5.24) and (5.23) imply that one needs only the center c in order to determine the median-interpolating polynomial p_{med}. Oswald suggested the following approximation for c.

Lemma 5.2 *Let $v = [v_{-1}, v_0, v_1]$ be an arbitrary triple with $\triangle^2 v \neq 0$.*

a) We have a canonical representation

$$v = v_0 + \frac{\triangle^2 v}{2}[1+\gamma, 0, 1-\gamma], \qquad \gamma = \frac{v_{-1} - v_1}{\triangle^2 v}. \tag{5.25}$$

b) The center c of the polynomial p_{med} associated with v is a continuous, piecewise smooth (w.r.t. $-\infty < -\frac{5}{2} < -\frac{3}{2} < -\frac{1}{2} < \frac{1}{2} < \frac{3}{2} < \frac{5}{2} < +\infty$) strictly monotone function of γ. More precisely, its inverse function $\gamma = \gamma(c)$ is explicitly given by

$$\gamma(c) = \frac{32c + \epsilon_{-2} - \epsilon_2}{32 + \epsilon_{-2} - 2\epsilon_0 + \epsilon_2}. \tag{5.26}$$

Proof. The first part follows from (5.20), and the second part - from (5.23). □

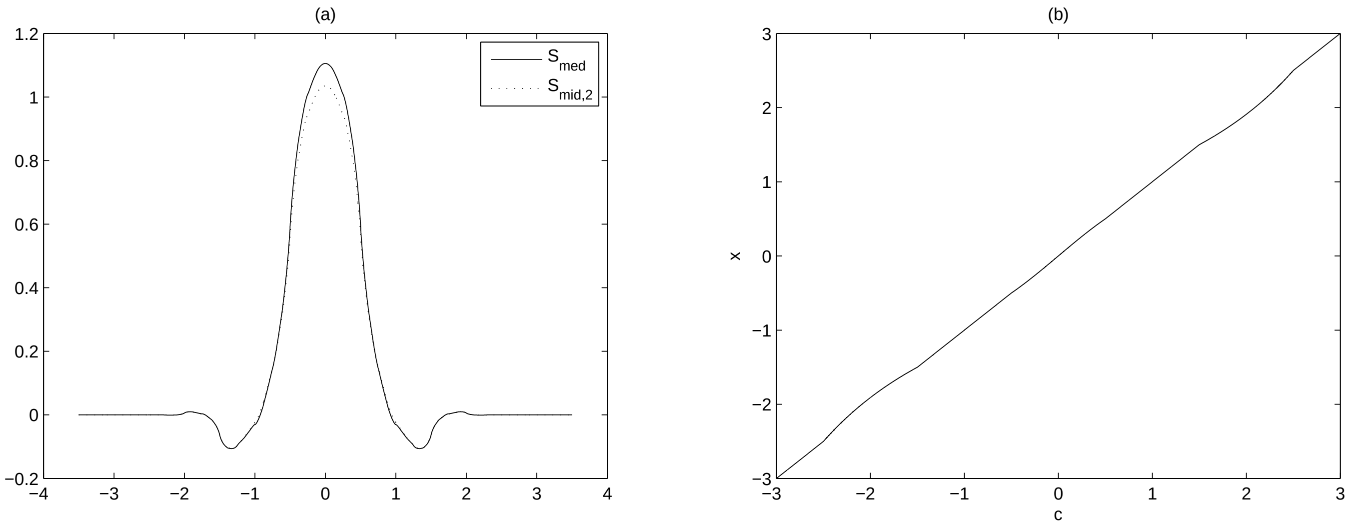

Figure 7: $S_{mid,2}$ vs. S_{med}: (a) limit functions on Dirac sequence (b)γ as a function on c

Remark 5.3 *Note that, γ is the center of the corresponding polynomial p_{mid}. Due to* (5.22)*, p_{med} and p_{mid}, and thus S_{med} and $S_{mid,2}$ are closely related (they even coincide, unless $\frac{v_{-1}-v_1}{\triangle^2 v} \in (-5/2,-3/2)\cup(-1/2,1/2)\cup(3/2,5/2)$). Hence, we can think of the median-interpolating subdivision scheme as slightly perturbed midpoint-interpolating subdivision scheme.*

Using (5.26), we can derive the inverse formula:

$$c(\gamma) = \begin{cases} 0, & \gamma = 0, \\ \frac{4-\sqrt{16-15\gamma^2}}{2\gamma}, & \gamma \in (-\frac{1}{2},0)\cup(0,\frac{1}{2}), \\ \frac{4(\gamma-1)+\sqrt{33\gamma^2-30\gamma+1}}{2(1+\gamma)}, & \gamma \in (\frac{3}{2},\frac{5}{2}), \\ \frac{4(\gamma+1)-\sqrt{33\gamma^2+30\gamma+1}}{2(1-\gamma)}, & \gamma \in (-\frac{5}{2},-\frac{3}{2}), \\ \gamma, & \textit{otherwise}. \end{cases}$$

For the refined medians, (5.22) gives

$$\begin{aligned} v_0^1 &= p_{med}(-\frac{1}{2}) + \frac{a}{16}\tilde{\epsilon}_{-1/2}, \qquad \tilde{\epsilon}_{-1/2} := (1-4(-1-2c)^2)_+, \\ v_1^1 &= p_{med}(\frac{1}{2}) + \frac{a}{16}\tilde{\epsilon}_{1/2}, \qquad \tilde{\epsilon}_{1/2} := (1-4(1-2c)^2)_+. \end{aligned} \tag{5.27}$$

Hence, using (5.22), (4.36) and (5.24)

$$\begin{aligned} v_0^1 &= p_{mid}(-\frac{1}{2}) - \alpha_{0,0}\delta = \frac{5}{32}v_{-1} + \frac{15}{16}v_0 - \frac{3}{32}v_1 - \alpha_{0,0}\delta, \\ v_1^1 &= p_{mid}(\frac{1}{2}) - \alpha_{1,0}\delta = -\frac{3}{32}v_{-1} + \frac{15}{16}v_0 + \frac{5}{32}v_1 - \alpha_{1,0}\delta, \end{aligned} \tag{5.28}$$

whereas

$$\alpha_{0,0} := \frac{30\epsilon_0 + 5\epsilon_{-2} - 3\epsilon_2 - 8\tilde{\epsilon}_{-1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)}, \qquad \alpha_{1,0} := \frac{30\epsilon_0 - 3\epsilon_{-2} + 5\epsilon_2 - 8\tilde{\epsilon}_{1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)}. \tag{5.29}$$

The second index stays for the stencil, i.e., due to shift-invariance the corresponding formulas for v^1_{2k} and v^1_{2k+1} will be the same as (5.28) with $\alpha_{i,k}, i = 0, 1$ instead of $\alpha_{i,0}$.

The median-interpolating subdivision scheme is convergent (see [DY00] for proof), but there are no results about stability, yet. Later in this paper we will provide strong numerical evidences that the scheme is stable.

6 Stability analysis on a univariate nonlinear subdivision

The following result is contained in the Bachelor's thesis of Christian Kühn:

Theorem 6.1 (Christian Kühn, see [Küh05]) *Let $S : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ be a (nonlinear) subdivision scheme. Decompose S into the 'sum' of a linear subdivision scheme S_{lin} and a nonlinear subdivision scheme S_{nlin}, in particular $Sv^j = S_{lin}v^j + S_{nlin}v^j$. Let $\{v^0, d^1, \ldots, d^J\}$ and $\{\tilde{v}^0, \tilde{d}^1, \ldots, \tilde{d}^J\}$ be two multiresolution decompositions for $v^J, \tilde{v}^J \in l_\infty(\mathbb{Z})$. Let $v^{j+1} := Mv^j = Sv^j + d^{j+1}$, where we denote by M the above multiresolution. Assume that for some nonnegative constants $C_0, C_1, K \in \mathbb{R}$, some $\rho \in (0, 1)$, finite $k, n \in \mathbb{N}$ and all $j \in \mathbb{N}$:*

$$\|S_{lin}\| \leq 1 \tag{6.1}$$

$$\|S_{nlin}v^j - S_{nlin}\tilde{v}^j\| \leq C_0\|\triangle^k(v^j - \tilde{v}^j)\| \tag{6.2}$$

$$\|\triangle^k(v^{j+n} - \tilde{v}^{j+n})\| \leq \rho\|\triangle^k(v^j - \tilde{v}^j)\| + C_1\sum_{i=1}^{n}\|\triangle^k(d^{j+i} - \tilde{d}^{j+i})\| \tag{6.3}$$

$$\|\triangle^k(v^{j+1} - \tilde{v}^{j+1})\| \leq K(\|\triangle^k(v^j - \tilde{v}^j)\| + \|\triangle^k(d^{j+1} - \tilde{d}^{j+1})\|). \tag{6.4}$$

Then one has the following inequality

$$\|v^J - \tilde{v}^J\| \leq C_2\|v^0 - \tilde{v}^0\| + C_3\sum_{j=1}^{J}\|d^j - \tilde{d}^j\| \tag{6.5}$$

with constants C_2, C_3 which depend on k, n, ρ, C_0, C_1 and K but not on J. In particular the multiresolution M associated to the subdivision scheme S is stable.

It is an attempt for generalizing the proof of the stability of PPH given by Amat and Liandrat [AL05], for a larger class of subdivision schemes (those that satisfy conditions (6.1)-(6.4)).

At first glance there are two main problems with Theorem 6.1:

- $\|S_{lin}\| \leq 1$

In section 4.2, we showed that in the linear case $S \equiv S_{lin}$, uniform convergence implies stability, and that S is uniformly convergent only if it reproduces constants. The latter means that $\|S\| \geq 1$. Moreover $\|S\| = 1$ if and only if all the non-trivial coefficients of S are positive. Therefore, Theorem 6.1 doesn't work as a necessary condition for stability even in the linear case. E.g., since (4.36) gives $\|S_{mid,2}\| = 19/16 > 1$, it does not apply to many schemes of interest.

- The nonuniqueness of the splitting of S into a linear and a nonlinear part.

There are many ways of splitting an operator S. For example, one could always take the trivial splitting $S_{lin} = \mathbf{0}$ and $S_{nlin} = S$. This makes the theorem dubious, since one can never know which splitting is the most optimal one for a specific scheme.

These two obstacles are the main reason for putting efforts into improving the above theorem and finding the *optimal* conditions for it, i.e., those conditions that will (eventually) lead to an "if and only if" statement for stability. A step in this direction is the following

Theorem 6.2 *Let $S : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ be a stationary (nonlinear) subdivision scheme. Let $\{v^0, d^1, \ldots, d^J\}$ and $\{\tilde{v}^0, \tilde{d}^1, \ldots, \tilde{d}^J\}$ be two arbitrary multiresolution decompositions for $v^J, \tilde{v}^J \in l_\infty(\mathbb{Z})$. Let $v^{j+1} := Mv^j = Sv^j + d^{j+1}$, where we denote by M the above multiresolution. Assume that for some nonnegative constants $C_0, C_1 \in \mathbb{R}$, some $\rho \in (0,1)$, and finite $k, n \in \mathbb{N}$:*

$$\|Sv^0 - S\tilde{v}^0\| \leq \|v^0 - \tilde{v}^0\| + C_0\|\triangle^k(v^0 - \tilde{v}^0)\| \tag{6.6}$$

$$\|\triangle^k(v^n - \tilde{v}^n)\| \leq \rho\|\triangle^k(v^0 - \tilde{v}^0)\| + C_1 \sum_{i=1}^{n} \|d^i - \tilde{d}^i\|. \tag{6.7}$$

Then one has the following inequality

$$\|v^J - \tilde{v}^J\| \leq C_2\|v^0 - \tilde{v}^0\| + C_3 \sum_{j=1}^{J} \|d^j - \tilde{d}^j\| \tag{6.8}$$

with constants C_2, C_3 which depend on k, n, ρ, C_0 and C_1 but not on J. In particular the subdivision scheme S, and the multiresolution M associated to it are stable.

Proof. Since S is stationary, (6.6) and (6.7) give rise to

$$\|Sv^j - S\tilde{v}^j\| \leq \|v^j - \tilde{v}^j\| + C_0\|\triangle^k(v^j - \tilde{v}^j)\|$$

$$\|\triangle^k(v^{j+n} - \tilde{v}^{j+n})\| \leq \rho\|\triangle^k(v^j - \tilde{v}^j)\| + C_1 \sum_{i=1}^{n} \|d^{j+i} - \tilde{d}^{j+i}\|$$

for every $j \in \mathbb{N}$. Fix $0 \leq j \leq J-1$.

$$\begin{aligned}\|v^{j+1} - \tilde{v}^{j+1}\| &= \|Sv^j + d^{j+1} - (S\tilde{v}^j + \tilde{d}^{j+1})\| \leq \|Sv^j - S\tilde{v}^j\| + \|d^{j+1} - \tilde{d}^{j+1}\| \\ &\leq \|v^j - \tilde{v}^j\| + C_0\|\triangle^k(v^j - \tilde{v}^j)\| + \|d^{j+1} - \tilde{d}^{j+1}\|.\end{aligned}$$

Applying this result iteratively J times we derive

$$
\begin{aligned}
\|v^J - \tilde{v}^J\| &\le \|v^{J-1} - \tilde{v}^{J-1}\| + C_0\|\triangle^k(v^{J-1} - \tilde{v}^{J-1})\| + \|d^J - \tilde{d}^J\| \\
&\le \|v^0 - \tilde{v}^0\| + \underbrace{C_0 \sum_{i=0}^{J-1} \|\triangle^k(v^i - \tilde{v}^i)\|}_{:=A} + \sum_{i=1}^{J} \|d^i - \tilde{d}^i\|. \qquad (6.9)
\end{aligned}
$$

Now to prove our theorem, it suffices to show that A can be estimated by the expression in the right-hand side of (6.8). Then from (6.7) it follows that

$$
\|\triangle^k(v^i - \tilde{v}^i)\| \le \rho\|\triangle^k(v^{i-n} - \tilde{v}^{i-n})\| + C_1 \sum_{t=0}^{n-1} \|d^{i-t} - \tilde{d}^{i-t}\|.
$$

Let $s := \lfloor i/n \rfloor$. Then by induction

$$
\|\triangle^k(v^i - \tilde{v}^i)\| \le \rho^s\|\triangle^k(v^{i-sn} - \tilde{v}^{i-sn})\| + C_1 \sum_{r=0}^{s-1} \rho^r \sum_{t=rn}^{(r+1)n-1} \|d^{i-t} - \tilde{d}^{i-t}\|, \qquad (6.10)
$$

and after summation and using $\rho < 1$

$$
A \le C(\rho)\Big(\sum_{j=0}^{n-1} \|\triangle^k(v^j - \tilde{v}^j)\| + \sum_{i=1}^{J} \|d^i - \tilde{d}^i\|\Big).
$$

To estimate $\|\triangle^k(v^j - \tilde{v}^j)\|$ for $j = 1, \ldots, n-1$, we use (6.6):

$$
\|Sv^j - S\tilde{v}^j\| \le \|v^j - \tilde{v}^j\| + C_0\|\triangle^k(v^j - \tilde{v}^j)\| \le (2^k C_0 + 1)\|v^j - \tilde{v}^j\|,
$$

which, applied j times, gives rise to

$$
\begin{aligned}
\|\triangle^k(v^j - \tilde{v}^j)\| &\le 2^k\|Sv^{j-1} - S\tilde{v}^{j-1}\| + 2^k\|d^j - \tilde{d}^j\| \\
&\le 2^k(2^k C_0 + 1)\|v^{j-1} - \tilde{v}^{j-1}\| + 2^k\|d^{j-1} - \tilde{d}^{j-1}\| \\
&\le 2^k(2^k C_0 + 1)^j\|v^0 - \tilde{v}^0\| + 2^k \sum_{i=0}^{j-1} (2^k C_0 + 1)^i \|d^{j-i} - \tilde{d}^{j-i}\|.
\end{aligned}
$$

Thus,

$$
\|\triangle^k(v^j - \tilde{v}^j)\| \le 2^k(2^k C_0 + 1)^{n-1}\|v^0 - \tilde{v}^0\| + 2^k(2^k C_0 + 1)^{n-2} \sum_{i=1}^{n-1} \|d^i - \tilde{d}^i\|. \qquad (6.11)
$$

Combining (6.9), (6.10), (6.11) and

$$
\sum_{i=0}^{J-1} \rho^s \le \sum_{i=0}^{\infty} \rho^s = \sum_{i=0}^{\infty} \rho^{\lfloor i/n \rfloor} = n \sum_{i=0}^{\infty} \rho^i = \frac{n}{1-\rho}
$$

we derive

$$
\begin{aligned}
\|v^J-\tilde v^J\| \le{}& \|v^0-\tilde v^0\| + C_0\sum_{i=0}^{J-1}\rho^s\Big(2^k(2^kC_0+1)^{n-1}\|v^0-\tilde v^0\| + 2^k(2^kC_0+1)^{n-2}\sum_{j=1}^{n-1}\|d^j-\tilde d^j\|\Big)\\
&+C_1\sum_{i=1}^{J}\sum_{r=0}^{s-1}\rho^r\sum_{t=rn}^{(r+1)n-1}\|d^{i-t}-\tilde d^{i-t}\| + \sum_{i=1}^{J}\|d^i-\tilde d^i\|\\
\le&\Big(1+\frac{2^kC_0(2^kC_0+1)^{n-1}n}{1-\rho}\Big)\|v^0-\tilde v^0\| + C_3\sum_{j=1}^{J}\|d^j-\tilde d^j\|.
\end{aligned}
$$

C_3 is finite and doesn't depend on J, because one can easily check that for any fixed $1\le j\le J$ the coefficient in front of $\|d^j-\tilde d^j\|$ is of the same type as the coefficient in front of $\|v^0-\tilde v^0\|$, i.e., a finite sum of geometric series with respect to ρ times some uniformly bounded constants. Hence, for every j we can do the same trick as for the coefficient in front of $\|v^0-\tilde v^0\|$ and declare the maximal of them to be C_3. □

Remark 6.3 *During the proof of Theorem 6.2 we have obtained an upper bound for the constant* C_2*, namely*

$$C_2 \le 1+\frac{2^kC_0(2^kC_0+1)^{n-1}n}{1-\rho}. \tag{6.12}$$

Remark 6.4 *If we set all the details* $\{d^j\}$ *and* $\{\tilde d^j\}$ *to be zero, Theorem 6.2 gives stability on the subdivision scheme* S *itself.*

Remark 6.5 *If either* S *is linear or* $n=1$*, stability of the subdivision scheme implies stability of the corresponding multiresolution.*

This is obvious due to the following proposition.

Proposition 6.6 *If a stationary linear subdivision scheme* S *is stable, then so is the multiresolution* M *associated to it.*

Proof. Since S is stable, there exists $C_0\in\mathbb{R}_+$, such that for any $J\in\mathbb{N}$ and any $v^0,\tilde v^0\in l_\infty(\mathbb{Z})$

$$\|S^Jv^0-S^J\tilde v^0\|\le C_0\|v^0-\tilde v^0\|.$$

By induction it follows that

$$M^Jv^0 = S^Jv^0+\sum_{j=1}^{J}S^{J-j}d^j.$$

Therefore

$$\|M^Jv^0-M^J\tilde v^0\|\le\|S^Jv^0-S^J\tilde v^0\|+\sum_{j=1}^{J}\|S^{J-j}d^j-S^{J-j}\tilde d^j\|\le C_0\Big(\|v^0-\tilde v^0\|+\sum_{j=1}^{J}\|d^j-\tilde d^j\|\Big).$$

□

On the other hand, Remark 6.5 cannot be extended to a nonlinear subdivision scheme S and $n \geq 2$, because

$$M^2 v^0 = S(Sv^0 + d^1) + d^2 \neq S^2 v^0 + Sd^1 + d^2.$$

Remark 6.7 *Theorem 6.2 is a generalization of Theorem 6.1.*

Indeed, Theorem 6.1 is a corollary of Theorem 6.2 due to the following proposition.

Proposition 6.8 *Let M be a multiresolution, associated to a stationary subdivision rule S that satisfies* (6.1)-(6.4)*. Then M satisfies* (6.6) *and* (6.7)*, as well.*

Proof. Combining (6.1) and (6.2) and using triangle inequality for the norm, we have:

$$\begin{aligned}\|Sv^j - S\tilde{v}^j\| &= \|S_{lin}v^j + S_{nlin}v^j - (S_{lin}\tilde{v}^j + S_{nlin}\tilde{v}^j)\| \leq \|S_{lin}(v^j - \tilde{v}^j)\| + \|S_{nlin}v^j - S_{nlin}\tilde{v}^j\| \\ &\leq \|S_{lin}\|\|v^j - \tilde{v}^j\| + C_0\|\triangle^k(v^j - \tilde{v}^j)\| \leq \|v^j - \tilde{v}^j\| + C_0\|\triangle^k(v^j - \tilde{v}^j)\|.\end{aligned}$$

Now taking into account that S is stationary, we obtain that for every $v^0, \tilde{v}^0 \in l_\infty(\mathbb{Z})$

$$\|Sv^0 - S\tilde{v}^0\| \leq \|v^0 - \tilde{v}^0\| + C_0\|\triangle^k(v^0 - \tilde{v}^0)\|,$$

which is exactly (6.6).

Hence (6.1) $\cup$ (6.2)$\Rightarrow$(6.6).

Again, since S is stationary and $\|\triangle^k d\| \leq 2^k\|d\|, \forall d \in l_\infty(\mathbb{Z})$, we have that (6.3)$\Rightarrow$(6.7). □

On the other hand, (4.28) and (4.30) give that any stable linear subdivision scheme satisfy (6.6) and (6.7) with $k = 1$. Therefore, our theorem is "stronger" than Kühn's one.

The proof of Theorem 6.2 shows that, analogously to the multivariate linear case (see (4.39)), in the stability analysis on univariate nonlinear subdivision we can increase the level of abstractness and use more general functions than the infinity norm of the k-th order divided difference.

Definition 6.9 *Let $D : l_\infty(\mathbb{Z}) \to [0, \infty)$ be bounded, i.e., there exists $B > 0$ such that $D(v) \leq B\|v\|, \forall v \in l_\infty(\mathbb{Z})$. We say that S is D-**Lipschitz contractive**, if there exists $C > 0$, $\rho \in (0, 1)$ and $n \in \mathbb{Z}_+$ such that for all $v, \tilde{v} \in l_\infty(\mathbb{Z})$*

$$\|Sv - S\tilde{v}\| \leq \|v - \tilde{v}\| + CD(v - \tilde{v}), \tag{6.13}$$

$$D(S^n v - S^n \tilde{v}) \leq \rho D(v - \tilde{v}). \tag{6.14}$$

Corollary 6.10 *A stationary subdivision scheme S is stable, if S is D-Lipschitz contractive for some $D : l_\infty(\mathbb{Z}) \to [0, \infty)$.*

6.1 Analysis on conditions (6.6) and (6.7)

In this section we will try to analyze (6.6) and (6.7). We will prove that k from theorem 6.2 cannot be bigger than the order of polynomial reproduction of the subdivision scheme S. We will prove that (6.6) with $k \le 2$ always holds in case of a linear subdivision scheme S with order of polynomial reproduction at least 2. Finally, we will give a counter example of a stable linear scheme S that reproduces polynomials up to degree 2, for which (6.6) doesn't hold for $k = 3$. With the help of these results, we will conclude that (6.6) and (6.7) are independent, i.e. no one implies the other. For example, (6.1)-(6.4) are not independent, as (6.4) follows from (6.1) and (6.2).

Although the problem about stability of linear schemes is completely solved, and Theorem 6.2 is expected to help in the nonlinear case, a reasonable test for it is exactly the linear case. Therefore, in this section S will be a linear subdivision scheme.

Lemma 6.11 *Let $k \in \mathbb{N}$ and $v \in l_\infty(\mathbb{Z})$. Then $\triangle^k v \equiv 0$ if and only if for any $j \in \mathbb{N}$ there exists $p \in \Pi_{k-1}(\mathbb{R})$ depending on j, such that $v = p^j$, where $\Gamma^j = 2^{-j}\mathbb{Z}$.*

Proof. $\Leftarrow$) Fix $j \in \mathbb{N}$. Let $v = p^j$ for some $p \in \Pi_{k-1}(\mathbb{R})$. Then from the standard course in Numerical Methods (see[Boy98]) we have that

$$\triangle^k v_i = \triangle^k (p^j)_i = k! 2^{-jk} p[x_i^j, \dots, x_{i+k}^j] = 2^{-kj} p^{(k)}(\xi) = 0$$

where $p[x_0, \dots, x_n]$ denotes the divided difference of p with respect to the nodes $x_0, \dots, x_n$ and $x_i^j < \xi < x_{i+k}^j$. Hence $\triangle^k v \equiv 0$.

$\Rightarrow$) Fix $j \in \mathbb{Z}$. Fix $i \in \mathbb{Z}$. Then there exists a unique polynomial $p \in \Pi_k(\mathbb{R})$ such that

$$p((i+l)2^{-j}) = p(x_{i+l}^j) = v_{i+l}^j \quad \forall l = 0, 1, \dots, k.$$

From the Newton's formula (see [Boy98]) we have:

$$p(x) = \sum_{l=0}^{k} p[x_i^j, \dots, x_{i+l}^j](x - x_i^j)(x - x_{i+1}^j) \dots (x - x_{i+l-1}^j).$$

Let us calculate the leading coefficient a_k of $p(x)$:

$$a_k = p[x_i^j, \dots, x_{i+k}^j] = \frac{2^{jk} \triangle^k v_i}{k!} = 0.$$

Hence, $p \in \Pi_{k-1}(\mathbb{R})$.

Analogously, if $q, r \in \Pi_k(\mathbb{R})$ are the interpolation polynomials of v with respect to the nodes $x_{i+1}^j, \dots, x_{i+k+1}^j$ and $x_{i-1}^j, \dots, x_{i+k-1}^j$, then $q, r \in \Pi_{k-1}(\mathbb{R})$. Now

$$p(x_{i+1+l}^j) = v_{i+1+l} = q(x_{i+1+l}^j) \;\; \forall l = 0, \dots, k-1 \Longrightarrow p(x) \equiv q(x).$$

Analogously, $p(x) \equiv r(x)$, and applying induction in both the positive and the negative directions, we obtain that $p^j \equiv v$. $\square$

Remark 6.12 *The above lemma is not true for a general grid! Take the nodes* $x_0 = 0, x_1 = 2, x_2 = 3$, $v = (0, 2, 3) = x^0$, $\tilde{v} = (0, 2, 4)$. *Then* $\triangle^2 v = -1 \neq 0$ *and* $\triangle^2 \tilde{v} = 0$, *but* $\tilde{v} \neq p^0$ *for any* $p \in \Pi_1(\mathbb{R})$.

Proposition 6.13 *Let* S *be a linear stationary subdivision scheme* S *that satisfies* (6.6) *and* (6.7) *with order of polynomial reproduction* N. *Then*

$$0 \leq k \leq N. \tag{6.15}$$

Proof. Take $\tilde{v}^0 \equiv 0$. Hence for (6.7) we have

$$\|\triangle^k S^n v\| \leq \rho \|\triangle^k v\|. \tag{6.16}$$

Take $v = p^0$ for some $p \in \Pi_{k-1}(\mathbb{R})$. Then, from Lemma 6.11, it follows that $\triangle^k v \equiv 0$. (6.16) implies that $\triangle^k S^n v \equiv 0$ and again Lemma 6.11 gives $S^n v = q^1$ for some $q \in \Pi_{k-1}(\mathbb{R})$.

Therefore, S^n reproduces polynomials of degree $k-1$. Finally, due to Proposition 2.4 in [JZ04], and that for every $n \in \mathbb{N}$ the order of polynomial reproduction of S^n and S coincides, we obtain $k \leq N$. □

Having proved the above statement, one could dream of proving (6.6) with N instead of k. Then, since $\|\triangle^N v\| \leq 2^{N-k} \|\triangle^k v\|$, (6.6) will be true for any $k \leq N$, and thus (6.7) will imply (6.6). Unfortunately, as it is shown below, this is not true even for linear subdivision schemes and $k \geq 3$.

Lemma 6.14 *Let* S *be a linear stationary subdivision scheme with order of polynomial reproduction at least 2. Then, there exists a linear scheme* $\hat{S}$ *with positive coefficients, such that*

$$Sp^j = \hat{S}p^j$$

for all $p \in \Pi_1(\mathbb{R})$ *and all* $j \in \mathbb{N}$, *where* $\Gamma^j = 2^{-j}\mathbb{Z}$.

Proof. Since S reproduces polynomials of degree 1 $S(x)^0 = (x - c)^1$ for some $c \in \mathbb{R}$. Take $\hat{S}$ to be:

$$\begin{aligned}
(\hat{S}v)_{2k} &= \{c\} v_{k-[c]-1} + (1 - \{c\}) v_{k-[c]} \\
(\hat{S}v)_{2k+1} &= \{c\} \frac{v_{k-[c]-1} + v_{k-[c]}}{2} + (1 - \{c\}) \frac{v_{k-[c]} + v_{k-[c]+1}}{2}
\end{aligned}$$

Obviously $\hat{S}$ has positive coefficients. Let $f = (x)^0$, i.e., $f_k = k \;\; \forall k \in \mathbb{Z}$. Then:

$$\begin{aligned}
(\hat{S}f)_{2k} &= \{c\}(k - [c] - 1) + (1 - \{c\})(k - [c]) = k - [c] - \{c\} = k - c = (Sf)_{2k} \\
(\hat{S}f)_{2k+1} &= \frac{f_{2k} + f_{2k+2}}{2} = k + \frac{1}{2} - c = (Sf)_{2k+1}.
\end{aligned}$$

Hence $\hat{S}f = Sf$. Due to the linearity of S, it follows that for every $p \in \Pi_1(\mathbb{R})$ $Sp^0 = \hat{S}p^0$ and, since both S and $\hat{S}$ are stationary, the Lemma is proved. □

Now we are ready to prove that for linear schemes, (6.6) with $k = 2$ always holds.

Proposition 6.15 *Let S be a linear stationary subdivision rule with order of polynomial reproduction at least 2. Then there exists $K \in \mathbb{N}$ such that for all $v \in l_\infty(\mathbb{Z})$*

$$\|Sv\| \leq \|v\| + K\|\triangle^2 v\|. \tag{6.17}$$

Proof. Let $\hat{S}$ be as in Lemma 6.14. Then $(S - \hat{S})\mathbf{1} = \mathbf{0}$ and Lemma 4.7 implies that there exists a bounded linear operator T such that for each $v \in l_\infty(\mathbb{Z})$

$$(S - \hat{S})v = T\triangle v.$$

Moreover, from $S(x)^0 = \hat{S}(x)^0$ and Lemma 6.11 it follows that $\mathbf{0} = (S - \hat{S})(x)^0 = T\triangle(x)^0 = T\mathbf{1}$.

Applying Lemma 4.7 one more time, there exists a bounded linear operator $\hat{T}$ with

$$(S - \hat{S})v = T\triangle v = \hat{T}\triangle^2 v$$

for every $v \in l_\infty(\mathbb{Z})$.

$$\begin{aligned} \|Sv\| &= \|\hat{S}v + (S - \hat{S})v\| \leq \|\hat{S}v\| + \|(S - \hat{S})v\| \\ &\leq \|\hat{S}\|\|v\| + \|\hat{T}\|\|\triangle^2 v\| \leq \|v\| + K\|\triangle^2 v\|. \end{aligned}$$

For the last step, we use that $\hat{T}$ is bounded and that since $\hat{S}\mathbf{1} = \mathbf{1}$ and $\hat{S}$ is positive, $\|\hat{S}\| = 1$. □

Remark 6.16 *The above statement is not true for $k \geq 3$. For example, take $S_{mid,2}$, fix $\epsilon > 0$ and let $v \in l_\infty(\mathbb{Z})$ be symmetric with respect to $-1/2$ (i.e., $v_{-n} = v_{n-1}, \forall n \in \mathbb{N}$) and such that*

$$v_i = -(i + \frac{1}{2})^2, \qquad i = -1, 0, 1.$$

Let $v_n \leq v_{n-1}, \forall n \in \mathbb{N}$ and $\|\triangle^3 v\| = \epsilon$. To achieve that, we can follow the following algorithm:

- Step 1*: Start with $n = 0$.*
- Step 2*: While ($\triangle^2 v_{n-1} < 0$) do $\left\{\triangle^2 v_n = \min(\triangle^2 v_{n-1} + \epsilon, 0); n = n + 1\right\}$.*
- Step 3*: While ($\triangle^1 v_n < 0$) do $\left\{\triangle^1 v_{n+1} = \triangle^1 v_{n+2} = \min(\triangle^1 v_n + \epsilon, 0); n = n + 2\right\}$.*

The case $n < 0$ follows by symmetry. Now let $A = A(\epsilon) := \|v\|$ and $w := A + v$. Then $w_i \geq 0, \forall i \in \mathbb{Z}$, $\|w\| = w_0 = A - \frac{1}{4}$, and $\|\triangle^3 w\| = \|\triangle^3 v\| = \epsilon$.
On the other hand,

$$\|S_{mid,2}w\| \geq |(S_{mid,2}w)_0| = |A + (S_{mid,2}v)_0| = A - \frac{1}{16}.$$

Hence,

$$\|S_{mid,2}w\| - \|w\| \geq \frac{3}{16} = \frac{3}{16\epsilon}\|\triangle^3 w\|,$$

and letting $\epsilon \to 0$ we see that (6.6) doesn't hold for $S_{mid,2}$ and $k = 3$ with uniformly bounded constant C_0.

Note that the above algorithm can be applied to any linear subdivision scheme that reproduces constants and has at least one negative coefficient! Thus, we can conclude that the only "meaningful" values of k are 1 and 2.

Corollary 6.17 (6.6) *does not imply* (6.7)*!*

Proof. In [AL05] it is shown that (6.7) with $k = 1$ does not hold for the PPH subdivision scheme for any $n \in \mathbb{N}$. On the other hand, PPH is a second-order perturbation of a linear subdivision scheme (see [AL05]). Hence, Proposition 6.15 is applicable, and (6.6) holds for $k = 1$. $\square$

Corollary 6.18 (6.7) *does not imply* (6.6)*!*

Proof. Follows directly from Remark 6.16 and Example 4.8 with $S = S_{mid,2}$, $k = 3$, and $n = 1$. $\square$

6.2 The Differential Approach

In this section we work only with dyadic schemes and we try to develop an algorithm for proving (6.7). In general, (6.7) is much harder to prove than (6.6). It is a generalization of the contracting property (4.28) from the linear case, which we interpreted as:
"*The spectral radius of the matrix, corresponding to the first derived scheme, is less than* 1".

Remark 5.1 indicates that the "matrix" approach may be extended to the nonlinear case as well. The main problem is, that due to nonlinearity, we are no longer interested in the behavior of the single sequence $\left\{\triangle^k S^J v^0\right\}_{J=0}^{\infty}$, but we are interested in the difference between this sequence and $\left\{\triangle^k S^J \tilde{v}^0\right\}_{J=0}^{\infty}$. In other words, we need to study the properties of the "derivative" of the derived scheme, rather than the properties of the scheme itself.
In order to be able to differentiate the k-th derived scheme, the quasilinear map Φ should be a function of $\triangle^k v$ instead of v. For $k = 1$ (5.10) and (5.15) show that this is true. Moreover, we can completely characterize the class of subdivision schemes S with the above property.

Lemma 6.19 *Let S be a (nonlinear) offset invariant subdivision scheme (i.e., $S(v + \mathbf{c}) = Sv + \mathbf{c}, \forall v \in l_\infty(\mathbb{Z}), c \in \mathbb{R}$), with quasilinear representation*

$$Sv = \Phi(v)v.$$

Then, there exists a quasilinear map $\tilde{\Phi}$ such that

$$Sv = \tilde{\Phi}(\triangle^1 v)v. \tag{6.18}$$

Proof. From the definition of quasilinearity it follows that the family of linear operators $\{\Phi(v) : v \in l_\infty(\mathbb{Z})\}$ is quasilinear with respect to S if and only if for every $v \in l_\infty(\mathbb{Z})$ $\Phi(v)$ has support not bigger than the one of S and the action of S and $\Phi(v)$ on v coincide. Let us define the relation $\sim$ in the space of all bounded real sequences by

$$v \sim w \Longleftrightarrow \exists c \in \mathbb{R} : v = w + \mathbf{c}. \tag{6.19}$$

Obviously this is an equivalence relation, and let us denote the equivalence class of a given sequence v by $[v]$. Then $\triangle^1 v = \triangle^1 w \Leftrightarrow w \in [v]$. Now, from each equivalence class in $l_\infty(\mathbb{Z})$ take one representative v_α and define

$$\tilde{\Phi}(\triangle^1 w) := \Phi(v_\alpha), \qquad \forall w \in [v_\alpha]. \tag{6.20}$$

Since S is offset invariant, (6.18) is satisfied. □

Assume that S is dyadic, finite, offset and shift invariant. Let $\Phi(v)$ and thus $\tilde{\Phi}(v)$ reproduces constants for any $v \in l_\infty(\mathbb{Z})$. (We can assume this, because the offset invariance of S implies $S\mathbf{1} = \mathbf{1}$.) Proposition 4.3 gives rise to

$$\triangle^1 S v = \tilde{\Phi}_1(\triangle^1 v)\triangle^1 v, \qquad v \in l_\infty(\mathbb{Z}). \tag{6.21}$$

Let us denote by $x^j := \triangle^1 S^j v$. We have

$$\tilde{\Phi}_1(x^{j-1})x_i^{j-1} = \sum_{l\in I} a_{i-2l}(x^{j-1})x_l^{j-1}, \tag{6.22}$$

where I is the support of S and a_{i-2l} is a function only of $x_l^{j-1}, l \in I$.
Assume for simplicity that $\{a_{i-2l}\}$ are differentiable. Then

$$\mathbf{d}\Big(a_{i-2l}(x^{j-1})\Big) = \sum_{s\in I} \frac{\partial a_{i-2l}}{\partial x_s^{j-1}}(x^{j-1})\mathbf{d}x_s^{j-1}. \tag{6.23}$$

Hence,

$$\begin{aligned}
\mathbf{d}x_i^j &= \mathbf{d}\Big(\tilde{\Phi}_1(x^{j-1})x_i^{j-1}\Big) = \mathbf{d}\Big(\sum_{l\in I} a_{i-2l}(x^{j-1})x_l^{j-1}\Big)\\
&= \sum_{l\in I}\bigg(\mathbf{d}\Big(a_{i-2l}(x^{j-1})\Big)x_l^{j-1} + a_{i-2l}(x^{j-1})\mathbf{d}x_l^{j-1}\bigg)\\
&= \sum_{l\in I}\bigg(\sum_{s\in I}\frac{\partial a_{i-2l}}{\partial x_s^{j-1}}(x^{j-1})x_l^{j-1}\mathbf{d}x_s^{j-1} + a_{i-2l}(x^{j-1})\mathbf{d}x_l^{j-1}\bigg)\\
&= \sum_{l\in I}\bigg(\sum_{s\in I}\frac{\partial a_{i-2s}}{\partial x_l^{j-1}}(x^{j-1})x_s^{j-1} + a_{i-2l}(x^{j-1})\bigg)\mathbf{d}x_l^{j-1}\\
&= \sum_{l\in I} t_{i,l}(x^{j-1})\mathbf{d}x_l^{j-1}.
\end{aligned}$$

We showed that there exists a linear, local (but not shift invariant!) data dependent operator $T : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$, such that

$$\mathbf{d}\big(x^j\big) = T(x^{j-1})\mathbf{d}\big(x^{j-1}\big), \qquad \forall j \in \mathbb{N}. \tag{6.24}$$

Therefore, by recursion it follows that

$$\mathbf{d}\big(x^j\big) = T(x^{j-1})T(x^{j-2})\dots T(x^0)\mathbf{d}\big(x^0\big), \qquad \forall j \in \mathbb{N}. \tag{6.25}$$

Remark 6.20 *The functions $a_{i-2l} : l_\infty(\mathbb{Z}) \to \mathbb{R}$ and, thus the operator T do not depend on the quasilinear representation Φ and are uniquely determined by S!*

Now let $v^0, \tilde{v}^0 \in l_\infty(\mathbb{Z})$. Define $x(t)^j : [0,1] \to l_\infty(\mathbb{Z})$

$$x(t)^j := \triangle^1 S^j(tv^0 + (1-t)\tilde{v}^0). \tag{6.26}$$

Then,

$$\begin{aligned} \|\triangle^1(S^n v^0 - S^n\tilde{v}^0)\| &= \|x(1)^n - x(0)^n\| = \|\int_0^1 \mathbf{d}\big(x(t)^n\big)\| \\ &= \|\int_0^1 T(x(t)^{n-1})T(x(t)^{n-2})\dots T(x(t)^0)\mathbf{d}\big(x(t)^0\big)\| \\ &\le \sup_{v\in l_\infty(\mathbb{Z})} \|T(\triangle^1 S^{n-1}v)T(\triangle^1 S^{n-2}v)\dots T(\triangle^1 v)\|\|\triangle^1(v^0-\tilde{v}^0)\|. \end{aligned} \tag{6.27}$$

For the last result, we needed T to be only Lipschitz continuous, and thus the coefficients a_{i-2l} not to be differentiable, but just in the Sobolev space $W^1_\infty(\mathbb{R})$. Hence, we proved the following:

Theorem 6.21 *Let S be local, offset and shift invariant, stationary subdivision scheme with $a_{i-2l} \in W^1_\infty(\mathbb{R})$, where $\{a_{i-2l}\}$ are defined by (6.22). Let S_1 be the first derived scheme of S. Then a sufficient condition for (6.7) to hold for S with $k=1$ and $d^j = \tilde{d}^j \equiv 0$ is: the* ***joint spectral radius of** T **with respect to*** S_1 *is less than 1, i.e.,*

$$\rho(T, S_1) := \liminf_{j\to\infty} \sup_{v\in l_\infty(\mathbb{Z})} \|T(S_1^{j-1}v)T(S_1^{j-2}v)\dots T(v)\|^{1/j} < 1. \tag{6.28}$$

Furthermore, we can extend the above result also for the multiresolution M associated to S:

Theorem 6.22 *Let S be local, offset and shift invariant, stationary subdivision scheme with $a_{i-2l} \in W^1_\infty(\mathbb{R})$, where $\{a_{i-2l}\}$ are defined by (6.22). Then a sufficient condition for (6.7) to hold for the multiresolution operator M associated to S with $k=1$ is: the* ***joint spectral radius of** T **with respect to*** M_1 *is less than 1, i.e.,*

$$\rho(T, M_1) := \liminf_{j\to\infty} \sup_{v,d^1,\dots,d^{j-1}\in l_\infty(\mathbb{Z})} \|T(M_1^{j-1}v)T(M_1^{j-2}v)\dots T(v)\|^{1/j} < 1, \tag{6.29}$$

where $M_1^k v = S_1(M_1^{k-1}v) + \triangle^1 d^k$, $\forall k \le j-1$.

Proof. It is a standard fact in functional analysis that (6.29) is equivalent to existence of a constant $C > 1$ and $\rho \in (0,1)$ such that for arbitrary $j \ge 0$ one has

$$\|T(M_1^{j-1}v)T(M_1^{j-2}v)\dots T(v)\| \le C\rho^j. \tag{6.30}$$

Let $v^0, \tilde{v}^0 \in l_\infty(\mathbb{Z})$. Denote by $y^j := \triangle^1 M^j v$. Then (6.24) gives rise to

$$\mathbf{d}\big(y^j\big) = T(y^{j-1})\mathbf{d}\big(y^{j-1}\big) + \mathbf{d}\big(\triangle^1 d^j\big) \qquad \forall j \in \mathbb{N}, \tag{6.31}$$

and thus, by recursion

$$\mathbf{d}\big(y^j\big) = T(y^{j-1})T(y^{j-2})\dots T(y^0)\mathbf{d}\big(y^0\big) + T(y^{j-2})\dots T(y^0)\mathbf{d}\big(\triangle^1 d^j\big) + \dots + \mathbf{d}\big(\triangle^1 d^1\big). \qquad (6.32)$$

Now, define $y(t)^j := \triangle^1 M^j(tv^0 + (1-t)\tilde{v}^0)$ and, analogously to (6.27), using the additive property of the integral, (6.30), and that $x^0 \equiv y^0$ we derive

$$\begin{aligned}\|\triangle^1(M^n v^0 - M^n\tilde{v}^0)\| &\le \mu\|\triangle^1(v^0 - \tilde{v}^0)\| + C\sum_{j=1}^{n}\|\triangle^1(d^j - \tilde{d}^j)\| \\ &\le \mu\|\triangle^1(v^0 - \tilde{v}^0)\| + 2C\sum_{j=1}^{n}\|d^j - \tilde{d}^j\|.\end{aligned} \qquad (6.33)$$

Here $n := \min\{j \in \mathbb{Z} : C\rho^j < 1\}$ and $\mu := C\rho^n \in (0,1)$. □

Proposition 6.21 and Proposition 6.22 can be straightforwardly generalized for arbitrary $k \in \mathbb{Z}$ and r-adic subdivision scheme. Let us denote by Ω_k the class of nonlinear, shift-invariant subdivision schemes S that allow representation

$$Sv = \Phi(\triangle^k v)v \implies \triangle^k Sv = S_k\triangle^k v = \Phi(\triangle^k v)\triangle^k v =: \Psi(\triangle^k v),$$

where Φ is an operator-valued function, and Ψ has uniformly bounded generalized partial derivatives (i.e., $\Psi \in W^1_\infty$).

Corollary 6.23 ***(Spectral radius version of Theorem 6.2)*** *Let $S \in \Omega_k$ such that*

$$\|Sv - S\tilde{v}\| \le \|v - \tilde{v}\| + C_0\|\triangle^k(v - \tilde{v})\|, \qquad \forall v, \tilde{v} \in l_\infty(\mathbb{Z}).$$

Let T be the linear data dependent operator that satisfies

$$\mathbf{d}\big(\triangle^k Sv\big) = T(\triangle^k v)\mathbf{d}\big(\triangle^k v\big).$$

Then if S_k is the kth derived scheme of S and M_k the kth derived scheme of the corresponding multiresolution operator M

(i) S is stable if $\rho(T, S_k) < 1$.

(ii) M is stable if $\rho(T, M_k) < 1$.

Sometimes (PPH for example) the spectral radius of T

$$\rho(T) := \liminf_{j\to\infty} \sup_{(u^0,u^1,\dots,u^{j-1})\in(l_\infty(\mathbb{Z}))^j} \|T(u^{j-1})T(u^{j-2})\dots T(u^0)\|^{1/j} \qquad (6.34)$$

is smaller than 1 and we do not need to consider S_k or M_k. On the other hand, M_k depends on the details and

$$\rho(T, M_k) = \rho(T)$$

unless we put some restrictions on the d^j's. Section 2.2 suggests that, in general, the details cannot be arbitrary, but the question what is the admissible class of details for a given mutliresolution is still open and heavily depends on the corresponding subdivision scheme S.

7 Applications of Theorem 6.2

7.1 PPH

One of the easiest applications of the newly developed machinery is the PPH scheme. The beginning of the proof of Theorem 1 in [AL05] says that for any $v^0, \tilde{v}^0 \in l_\infty(\mathbb{Z})$

$$\|S_{PPH}v^0 - S_{PPH}\tilde{v}^0\| \leq \|v^0 - \tilde{v}^0\| + \frac{1}{4}\|\triangle^2(v^0 - \tilde{v}^0)\|, \tag{7.1}$$

and thus, (6.6) holds.

(5.15), together with the fact that the function

$$C_k(v) = \frac{\operatorname{sgn}(\triangle^2 v_{k-1}\triangle^2 v_k) + 1}{8} \frac{\triangle^2 v_k}{\triangle^2 v_{k-1} + \triangle^2 v_k}$$

defined in (5.14) is continuous and almost everywhere differentiable in $\mathbb{R}^2$ allows us to use the "differential" approach. For convenience, we will denote by $\triangle_k := \triangle^2 v_k$, by $\chi_k := \frac{\operatorname{sgn}(\triangle_{k-1}\triangle_k)+1}{2}$, and by $\tilde{\triangle}_k := (\triangle^2 S_{PPH}v)_k$. Note that χ_k is 1 for convex (concave) data, and 0 otherwise.
Direct computations give:

$$\tilde{\triangle}_{2k} = \frac{\chi_k}{2}\frac{\triangle_{k-1}\triangle_k}{\triangle_{k-1} + \triangle_k} \tag{7.2}$$

$$\tilde{\triangle}_{2k+1} = \frac{\triangle_k}{2} - \frac{\chi_k}{4}\frac{\triangle_{k-1}\triangle_k}{\triangle_{k-1} + \triangle_k} - \frac{\chi_{k+1}}{4}\frac{\triangle_k\triangle_{k+1}}{\triangle_k + \triangle_{k+1}}. \tag{7.3}$$

Therefore, for the entries of the differential matrix $T(v) = \{t_{i,j}(v)\}$ we obtain

$$\mathbf{d}(\tilde{\triangle}_{2k}) = \frac{\chi_k}{2}\frac{(\triangle_k)^2}{(\triangle_{k-1} + \triangle_k)^2}\mathbf{d}(\triangle_{k-1}) + \frac{\chi_k}{2}\frac{(\triangle_{k-1})^2}{(\triangle_{k-1} + \triangle_k)^2}\mathbf{d}(\triangle_k), \tag{7.4}$$

$$\begin{aligned}\mathbf{d}(\tilde{\triangle}_{2k+1}) =& \left(\frac{1}{2} - \frac{\chi_k}{4}\frac{(\triangle_{k-1})^2}{(\triangle_{k-1} + \triangle_k)^2} - \frac{\chi_{k+1}}{4}\frac{(\triangle_{k+1})^2}{(\triangle_k + \triangle_{k+1})^2}\right)\mathbf{d}(\triangle_k) \\ &- \frac{\chi_k}{4}\frac{(\triangle_k)^2}{(\triangle_{k-1} + \triangle_k)^2}\mathbf{d}(\triangle_{k-1}) - \frac{\chi_{k+1}}{4}\frac{(\triangle_k)^2}{(\triangle_k + \triangle_{k+1})^2}\mathbf{d}(\triangle_{k+1})\end{aligned} \tag{7.5}$$

Hence, for any $k \in \mathbb{Z}$ the only nontrivial entries of T are $t_{2k,k-1}, t_{2k,k}, t_{2k+1,k}, t_{2k+1,k-1}$ and $t_{2k+1,k+1}$.
Moreover,

$$0 \leq t_{2k,k-1}, t_{2k,k}, t_{2k+1,k} \leq \frac{1}{2}, \qquad -\frac{1}{4} \leq t_{2k+1,k-1}, t_{2k+1,k+1} \leq 0, \tag{7.6}$$

and

$$\sum_{i\in\mathbb{Z}} |t_{2k,k+i}| = \frac{\chi_k}{2}\left(\frac{(\triangle_k)^2}{(\triangle_{k-1} + \triangle_k)^2} + \frac{(\triangle_{k-1})^2}{(\triangle_{k-1} + \triangle_k)^2}\right) < \frac{1}{2} \tag{7.7a}$$

$$\begin{aligned}\sum_{i\in\mathbb{Z}} |t_{2k+1,k+i}| =& \frac{1}{2} - \frac{\chi_k}{4}\frac{(\triangle_{k-1})^2}{(\triangle_{k-1} + \triangle_k)^2} + \frac{\chi_k}{4}\frac{(\triangle_k)^2}{(\triangle_{k-1} + \triangle_k)^2} - \frac{\chi_{k+1}}{4}\frac{(\triangle_{k+1})^2}{(\triangle_k + \triangle_{k+1})^2} + \frac{\chi_{k+1}}{4}\frac{(\triangle_k)^2}{(\triangle_k + \triangle_{k+1})^2} \\ &< 1.\end{aligned} \tag{7.7b}$$

This is not enough for claiming stability of PPH, since the infinity norm of $T(v)$ is 1. Indeed, take $\epsilon > 0$, $\triangle_{k-1} = \triangle_{k+1} = \epsilon$, and $\triangle_k = 1 - \epsilon$. Then

$$\sum_{i \in \mathbb{Z}} |t_{2k+1,k+i}| = 1 - \epsilon \xrightarrow[\epsilon \to 0]{} 1.$$

On the other hand, it suffices to consider two iterations of T:

Proposition 7.1 *For any $v, w \in l_\infty(\mathbb{Z})$*

$$\|T(w)T(v)\| \leq \frac{3}{4}. \tag{7.8}$$

Proof. Let us, for simplicity, denote the entries of $T(v)$ and $T(w)$ by $\{t_{i,j}\}$ and $\{\tilde{t}_{i,j}\}$, respectively. Then

$$\|T(w)T(v)\| = \max_{0 \leq i \leq 3} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\tilde{t}_{4k+i,l} t_{l,j}| \leq \max_{0 \leq i \leq 3} \sum_{l \in \mathbb{Z}} |\tilde{t}_{4k+i,l}| \Big(\sum_{j \in \mathbb{Z}} |t_{l,j}| \Big). \tag{7.9}$$

For even $i = 2i'$ (7.9) and (7.7a) give

$$\sum_{l \in \mathbb{Z}} |\tilde{t}_{4k+i,l}| \Big(\sum_{j \in \mathbb{Z}} |t_{l,j}| \Big) \leq \sum_{l \in \mathbb{Z}} |\tilde{t}_{2(2k+i'),l}| \leq \frac{1}{2}, \tag{7.10}$$

while for odd $i = 2i' + 1$

$$\sum_{l \in \mathbb{Z}} |\tilde{t}_{4k+i,l}| \Big(\sum_{j \in \mathbb{Z}} |t_{l,j}| \Big) \leq |\tilde{t}_{2(2k+i')+1,2k+i'-1}| + \frac{1}{2} |\tilde{t}_{2(2k+i')+1,2k+i'}| + |\tilde{t}_{2(2k+i')+1,2k+i'+1}| \leq \frac{3}{4}. \tag{7.11}$$

For the last inequality we used the estimate

$$|t_{2k+1,k-1}| + \frac{1}{2} |t_{2k+1,k}| + |t_{2k+1,k+1}| \leq \frac{3}{4}.$$

Its proof follows directly from (7.6). □

Therefore, the PPH multiresolution satisfies conditions (6.6) and (6.7) with constants

$$k = 2; \qquad n = 2; \qquad \rho = \frac{3}{4},$$

and thus, is stable!

7.2 WENO

In this section we will sketch a proof of stability of WENO subdivision scheme. We will use the notation developed in Section 5.1. Due to practical reasons and the fact that S_{WENO} is not affine invariant, we use a weaker version of stability for WENO (see [CDM03]), namely we allow the constants C_2 and C_3 in (6.8) to depend in a continuous nondecreasing way on $\max\{\|v^0\|, \|\tilde{v}^0\|\}$.

Lemma 7.2 *Let $v^0, \tilde{v}^0 \in l_\infty(\mathbb{Z})$. Then*

$$\|S_{WENO}v^0 - S_{WENO}\tilde{v}^0\| \leq \|v^0 - \tilde{v}^0\| + C_0\|\triangle^1(v^0 - \tilde{v}^0)\|, \tag{7.12}$$

where C_0 depends in a continuous nondecreasing way on $\max\{\|v^0\|, \|\tilde{v}^0\|\}$.

Proof. In [CDM03] it is proved that S_{WENO} *continuously depends* on data, i.e., with respect to the notation in (5.10) $\|\Phi(v^0)-\Phi(\tilde{v}^0)\| \leq B\|v^0-\tilde{v}^0\|$, where B depends in a continuous nondecreasing way on $\max\{\|v^0\|, \|\tilde{v}^0\|\}$. Therefore,

$$\|\tilde{\Phi}(\triangle^1 v^0) - \tilde{\Phi}(\triangle^1 \tilde{v}^0)\| \leq C\|\triangle^1(v^0 - \tilde{v}^0)\|, \tag{7.13}$$

where, again, C depends in a continuous nondecreasing way on $\max\{\|v^0\|, \|\tilde{v}^0\|\}$.
Now, (5.10) and (7.13) give

$$\begin{aligned}\|S_{WENO}v^0 - S_{WENO}\tilde{v}^0\| &= \|\tilde{\Phi}(\triangle^1 v^0)v^0 - \tilde{\Phi}(\triangle^1 \tilde{v}^0)\tilde{v}^0\| \\ &\leq \|\tilde{\Phi}(\triangle^1 v^0)v^0 - \tilde{\Phi}(\triangle^1 v^0)\tilde{v}^0\| + \|\tilde{\Phi}(\triangle^1 v^0)\tilde{v}^0 - \tilde{\Phi}(\triangle^1 \tilde{v}^0)\tilde{v}^0\| \\ &\leq \|\tilde{\Phi}(\triangle^1 v^0)v^0 - \tilde{\Phi}(\triangle^1 v^0)\tilde{v}^0\| + C\|\tilde{v}^0\|\|\triangle^1(v^0 - \tilde{v}^0)\|.\end{aligned} \tag{7.14}$$

On the other hand, the linear schemes S_l, S_c, and S_r, defined in Section 5.1 reproduce constants, and due to (4.30) there exist C_l, C_c, C_r, such that

$$\begin{aligned}\|S_l v^0 - S_l\tilde{v}^0\| &\leq \|v^0 - \tilde{v}^0\| + C_l\|\triangle^1(v^0 - \tilde{v}^0)\|; \\ \|S_c v^0 - S_c\tilde{v}^0\| &\leq \|v^0 - \tilde{v}^0\| + C_c\|\triangle^1(v^0 - \tilde{v}^0)\|; \\ \|S_r v^0 - S_r\tilde{v}^0\| &\leq \|v^0 - \tilde{v}^0\| + C_r\|\triangle^1(v^0 - \tilde{v}^0)\|.\end{aligned}$$

Let $D := \max(C_l, C_c, C_r)$. Hence, if $\tilde{\Phi}(\triangle^1 v^0) = \alpha_l S_l + \alpha_c S_c + \alpha_r S_r$, with $\alpha_l, \alpha_c, \alpha_r \geq 0$ and $\alpha_l + \alpha_c + \alpha_r = 1$, then

$$\begin{aligned}\|\tilde{\Phi}(\triangle^1 v^0)v^0 - \tilde{\Phi}(\triangle^1 v^0)\tilde{v}^0\| &\leq \alpha_l\|S_l v^0 - S_l\tilde{v}^0\| + \alpha_c\|S_c v^0 - S_c\tilde{v}^0\| + \alpha_r\|S_r v^0 - S_r\tilde{v}^0\| \\ &\leq \|v^0 - \tilde{v}^0\| + D\|\triangle^1(v^0 - \tilde{v}^0)\|.\end{aligned} \tag{7.15}$$

Finally, combining (7.14) and (7.15) we derive (7.12) with $C_0 := D + C\|\tilde{v}^0\|$. □

(5.9) shows that $\triangle^1 S_{WENO}v = (\alpha_l S_l^1 + \alpha_c S_c^1 + \alpha_r S_r^1)\triangle^1 v =: S_{WENO,1}\triangle^1 v$. For the first derived schemes of S_l, S_c, and S_r we have the explicit formulas

$$\begin{aligned}(S_l^1 v)_{2k} &= -\frac{1}{16}v_{k-2} + \frac{1}{4}v_{k-1} + \frac{5}{16}v_k & (S_l^1 v)_{2k+1} &= \frac{1}{16}v_{k-2} - \frac{1}{4}v_{k-1} + \frac{11}{16}v_k \\ (S_c^1 v)_{2k} &= \frac{1}{16}v_{k-1} + \frac{1}{2}v_k - \frac{1}{16}v_{k+1} & (S_c^1 v)_{2k+1} &= -\frac{1}{16}v_{k-1} + \frac{1}{2}v_k + \frac{1}{16}v_{k+1} \\ (S_r^1 v)_{2k} &= \frac{11}{16}v_k - \frac{1}{4}v_{k+1} + \frac{1}{16}v_{k+2} & (S_r^1 v)_{2k+1} &= \frac{5}{16}v_k + \frac{1}{4}v_{k+1} - \frac{1}{16}v_{k+2}.\end{aligned}$$

Let us denote by $x := \triangle^1 v$, and by $x^1 := \triangle^1 S_{WENO}v$. Then

$$\begin{aligned}x^1_{2k} &= -\frac{\alpha_{k,1}}{16}x_{k-2} + \frac{4\alpha_{k,1} + \alpha_{k,2}}{16}x_{k-1} + \frac{5\alpha_{k,1} + 8\alpha_{k,2} + 11\alpha_{k,3}}{16}x_k - \frac{\alpha_{k,2} + 4\alpha_{k,3}}{16}x_{k+1} + \frac{\alpha_{k,3}}{16}x_{k+2} \\ x^1_{2k+1} &= \frac{\alpha_{k,1}}{16}x_{k-2} - \frac{4\alpha_{k,1} + \alpha_{k,2}}{16}x_{k-1} + \frac{11\alpha_{k,1} + 8\alpha_{k,2} + 5\alpha_{k,3}}{16}x_k + \frac{\alpha_{k,2} + 4\alpha_{k,3}}{16}x_{k+1} - \frac{\alpha_{k,3}}{16}x_{k+2}.\end{aligned} \tag{7.16}$$

Note that the weights $\{\alpha_{k,i}\}_{i=1}^{3}$ depend only on $v_{k+j},\ j=-2,\dots 3$. Moreover, since WENO is offset invariant, Lemma 6.19 says that the weights are functions on $x_{k+j},\ j=-2,\dots,2$. Assume that for every i $\alpha_{k,i}\in W_{\infty}^{1}$ (for example this is true, when the cost function ϕ is differentiable). Then we can apply the "differential approach" with $k=1$ and for the "even" entries of the matrix T we obtain

$$\begin{aligned} t_{2k,k-2} = \frac{\partial\alpha_{k,1}}{\partial x_{k-2}}\Big[-\frac{x_{k-2}}{16}+\frac{x_{k-1}}{4}+\frac{5x_k}{16}\Big] + \frac{\partial\alpha_{k,2}}{\partial x_{k-2}}\Big[\frac{x_{k-1}}{16}+\frac{x_k}{2}-\frac{x_{k+1}}{16}\Big] \\ + \frac{\partial\alpha_{k,3}}{\partial x_{k-2}}\Big[\frac{11x_k}{16}-\frac{x_{k+1}}{4}+\frac{x_{k+2}}{16}\Big] - \frac{\alpha_{k,1}}{16} \end{aligned}$$

$$\begin{aligned} t_{2k,k-1} = \frac{\partial\alpha_{k,1}}{\partial x_{k-1}}\Big[-\frac{x_{k-2}}{16}+\frac{x_{k-1}}{4}+\frac{5x_k}{16}\Big] + \frac{\partial\alpha_{k,2}}{\partial x_{k-1}}\Big[\frac{x_{k-1}}{16}+\frac{x_k}{2}-\frac{x_{k+1}}{16}\Big] \\ + \frac{\partial\alpha_{k,3}}{\partial x_{k-1}}\Big[\frac{11x_k}{16}-\frac{x_{k+1}}{4}+\frac{x_{k+2}}{16}\Big] + \frac{4\alpha_{k,1}+\alpha_{k,2}}{16} \end{aligned}$$

$$\begin{aligned} t_{2k,k} = \frac{\partial\alpha_{k,1}}{\partial x_{k}}\Big[-\frac{x_{k-2}}{16}+\frac{x_{k-1}}{4}+\frac{5x_k}{16}\Big] + \frac{\partial\alpha_{k,2}}{\partial x_{k}}\Big[\frac{x_{k-1}}{16}+\frac{x_k}{2}-\frac{x_{k+1}}{16}\Big] \\ + \frac{\partial\alpha_{k,3}}{\partial x_{k}}\Big[\frac{11x_k}{16}-\frac{x_{k+1}}{4}+\frac{x_{k+2}}{16}\Big] + \frac{5\alpha_{k,1}+8\alpha_{k,2}+11\alpha_{k,3}}{16} \end{aligned}$$

$$\begin{aligned} t_{2k,k+1} = \frac{\partial\alpha_{k,1}}{\partial x_{k+1}}\Big[-\frac{x_{k-2}}{16}+\frac{x_{k-1}}{4}+\frac{5x_k}{16}\Big] + \frac{\partial\alpha_{k,2}}{\partial x_{k+1}}\Big[\frac{x_{k-1}}{16}+\frac{x_k}{2}-\frac{x_{k+1}}{16}\Big] \\ + \frac{\partial\alpha_{k,3}}{\partial x_{k+1}}\Big[\frac{11x_k}{16}-\frac{x_{k+1}}{4}+\frac{x_{k+2}}{16}\Big] - \frac{\alpha_{k,2}+4\alpha_{k,3}}{16}x_{k+1} \end{aligned}$$

$$\begin{aligned} t_{2k,k+2} = \frac{\partial\alpha_{k,1}}{\partial x_{k+2}}\Big[-\frac{x_{k-2}}{16}+\frac{x_{k-1}}{4}+\frac{5x_k}{16}\Big] + \frac{\partial\alpha_{k,2}}{\partial x_{k+2}}\Big[\frac{x_{k-1}}{16}+\frac{x_k}{2}-\frac{x_{k+1}}{16}\Big] \\ + \frac{\partial\alpha_{k,3}}{\partial x_{k+2}}\Big[\frac{11x_k}{16}-\frac{x_{k+1}}{4}+\frac{x_{k+2}}{16}\Big] + \frac{\alpha_{k,3}}{16}. \end{aligned}$$

The "odd" entries are similar due to symmetry and we will not write them explicitly.

$$\alpha_{k,1} = \frac{a_{k,1}}{a_{k,1}+a_{k,2}+a_{k,3}} \implies \frac{\partial\alpha_{k,1}}{\partial x_j} = \frac{\frac{\partial a_{k,1}}{\partial x_j}a_{k,2} + \frac{\partial a_{k,1}}{\partial x_j}a_{k,3} - a_{k,1}\frac{\partial a_{k,2}}{\partial x_j} - a_{k,1}\frac{\partial a_{k,3}}{\partial x_j}}{(a_{k,1}+a_{k,2}+a_{k,3})^2}.$$

Using (5.8) we derive

$$\begin{aligned} \frac{\partial a_{k,i}}{\partial x_j} &= \frac{-2\partial b_{k,i}/\partial x_j}{\epsilon+b_{k,i}}a_{k,i}, \quad \forall i=1,2,3,\ \forall j=k-2,\dots,k+2 \\ \implies \quad \frac{\partial\alpha_{k,1}}{\partial x_j} &= \frac{\Big[\frac{\partial b_{k,2}/\partial x_j}{\epsilon+b_{k,2}} - \frac{\partial b_{k,1}/\partial x_j}{\epsilon+b_{k,1}}\Big]2a_{k,1}a_{k,2} + \Big[\frac{\partial b_{k,3}/\partial x_j}{\epsilon+b_{k,3}} - \frac{\partial b_{k,1}/\partial x_j}{\epsilon+b_{k,1}}\Big]2a_{k,1}a_{k,3}}{(a_{k,1}+a_{k,2}+a_{k,3})^2}. \end{aligned}$$

All these formulas are very "ugly" and difficult to analyze. Therefore, it is practically impossible to estimate the joint spectral radius of T and to investigate stability of the associated multiresolution operator. However, since the WENO scheme is uniformly convergent, we have that

$$\triangle^1 S^j_{WENO} v \xrightarrow[j\to\infty]{} \mathbf{0}, \quad \forall v \in l_\infty(\mathbb{Z}).$$

This, together with our assumption that all the partial derivatives of the weights are uniformly bounded ($\alpha_{k,i} \in W^1_\infty$) implies that $T(S^j_{WENO}v) \xrightarrow[j\to\infty]{} S_{WENO,1}(S^j_{WENO}v)$ and thus

$$\rho(T, S_{WENO,1}) = \rho(S_{WENO,1}). \tag{7.17}$$

In [CDM03] Cohen, Dyn, and Matei have proven that

$$\sup_{u,w\in l_\infty(\mathbb{Z})} \|S_{WENO,1}(u)S_{WENO,1}(w)\| < 1,$$

which is enough for claiming stability of the WENO subdivision scheme.

7.3 Quadratic, dyadic median-interpolating subdivision scheme

The aim in this section is to sketch the proof of the stability of median subdivision in the dyadic case. We heavily rely on the explicit formulas from [Osw04], see also Section 5.3 above.

To set notation, let $x = \triangle^1 v$, $y = \triangle^1 S_{med} v$, i.e., $y = S_{med,1}x$, and $\delta = \triangle^2 v = \triangle^1 x$. Also, denote by $c = \{c_k\}$ the sequence of centers of median-interpolating polynomials associated with $\{v_{k-1}, v_k, v_{k+1}\}$, defined via (5.21). (Recall that $c_k = \infty$ is acceptable and corresponds to the case $\delta_{k-1} = 0$.) Finally, let $\gamma = \{\gamma_k\}$ be the sequence of centers of midpoint-interpolating polynomials associated with $\{v_{k-1}, v_k, v_{k+1}\}$ (see Lemma 5.2), i.e., $\gamma_k = -\frac{x_k + x_{k-1}}{\delta_{k-1}}$.

We will apply Theorem 6.2 with $k = 1$.

Lemma 7.3

$$\|S_{med}v - S_{med}\tilde{v}\| \le \|v - \tilde{v}\| + K\|\triangle^1(v - \tilde{v})\|,$$

for any $v, \tilde{v} \in l_\infty(\mathbb{Z})$, where K is a uniform bounded constant that does not depend on v and $\tilde{v}$.

Proof. The proof is in the appendix. ☐

For (6.7), we will apply the "differential" approach. From (5.28) and (5.29) it follows that for any $k \in \mathbb{Z}$

$$y_{2k} = \frac{1}{4}(x_{k-1} + x_k) + \delta_{k-1}\alpha_2(c_k), \tag{7.18}$$

$$y_{2k+1} = \frac{-3x_{k-1} + 22x_k - 3x_{k+1}}{32} + \delta_{k-1}\alpha_1(c_k) - \delta_k\alpha_0(c_{k+1}), \tag{7.19}$$

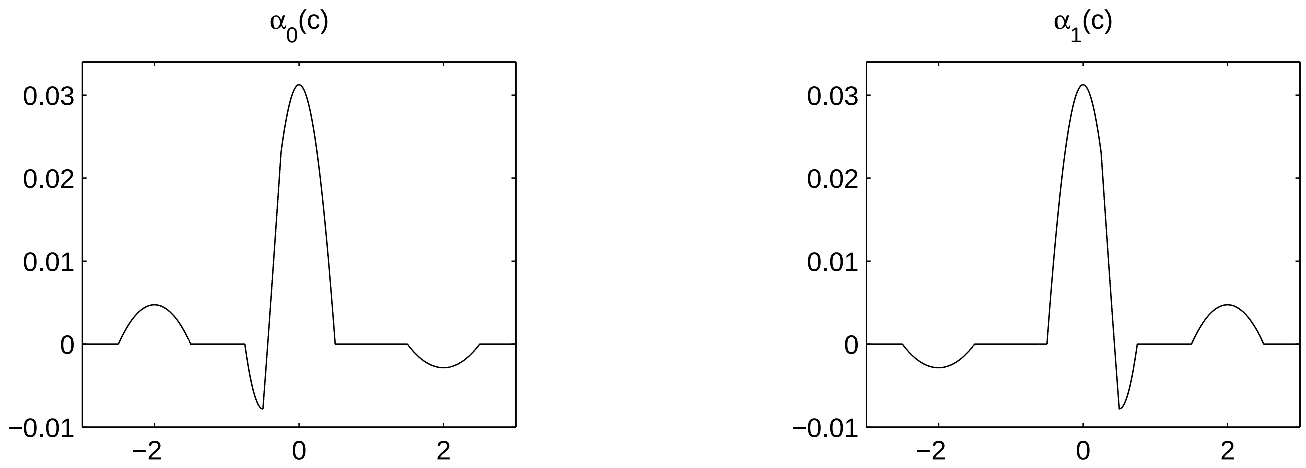

Figure 8: $\alpha_0(c)$ and $\alpha_1(c)$.

where $\alpha_l(c_k) := \alpha_{l,k}, l = 0, 1$ and $\alpha_2(c_k) := \alpha_0(c_k) - \alpha_1(c_k)$.
Hence, the nonlinear terms in (7.18) and (7.19) are of the form $\delta_{k-1}\alpha(c_k)$ for some piecewise smooth α. Formal differentiation gives

$$\begin{aligned}\mathbf{d}\big(\delta_{k-1}\alpha(c_k)\big) &= \alpha(c_k)\mathbf{d}\delta_{k-1} + \delta_{k-1}\alpha'(c_k)c'(\gamma_k)\mathbf{d}\gamma_k \\ &= \alpha(c_k)(\mathbf{d}x_k - \mathbf{d}x_{k-1}) + \delta_{k-1}\frac{\alpha'(c_k)}{\gamma'(c_k)}\mathbf{d}\gamma_k \\ &= \alpha(c_k)(\mathbf{d}x_k - \mathbf{d}x_{k-1}) + \delta_{k-1}\frac{\alpha'(c_k)}{\gamma'(c_k)}\Big(- \frac{\mathbf{d}x_k + \mathbf{d}x_{k-1}}{\delta_{k-1}} + \frac{x_k + x_{k-1}}{\delta_{k-1}^2}\mathbf{d}\delta_{k-1}\Big) \\ &= \Big(- \alpha(c_k) - \frac{\alpha'(c_k)}{\gamma'(c_k)} + \frac{\alpha'(c_k)}{\gamma'(c_k)}\gamma(c_k)\Big)\mathbf{d}x_{k-1} + \Big(\alpha(c_k) - \frac{\alpha'(c_k)}{\gamma'(c_k)} - \frac{\alpha'(c_k)}{\gamma'(c_k)}\gamma(c_k)\Big)\mathbf{d}x_k.\end{aligned}$$

This leads to the following explicit formulas for the entries $\{t_{k,l}\}$ of $T(x)$:

$$\mathbf{d}y_{2k} = \Big[\frac{1}{4} - \Big(\alpha_2 + \frac{\alpha_2'}{\gamma'}(1-\gamma)\Big)(c_k)\Big]\mathbf{d}x_{k-1} + \Big[\frac{1}{4} + \Big(\alpha_2 - \frac{\alpha_2'}{\gamma'}(1+\gamma)\Big)(c_k)\Big]\mathbf{d}x_k \tag{7.20}$$

$$\begin{aligned}\mathbf{d}y_{2k+1} =& \Big[- \frac{3}{32} - \Big(\alpha_1 + \frac{\alpha_1'}{\gamma'}(1-\gamma)\Big)(c_k)\Big]\mathbf{d}x_{k-1} + \Big[- \frac{3}{32} - \Big(\alpha_0 - \frac{\alpha_0'}{\gamma'}(1+\gamma)\Big)(c_{k+1})\Big]\mathbf{d}x_{k+1} \\ &+ \Big[\frac{11}{16} + \Big(\alpha_1 - \frac{\alpha_1'}{\gamma'}(1+\gamma)\Big)(c_k) + \Big(\alpha_0 + \frac{\alpha_0'}{\gamma'}(1-\gamma)\Big)(c_{k+1})\Big]\mathbf{d}x_k.\end{aligned} \tag{7.21}$$

Lemma 7.4 *For any $k \in \mathbb{Z}$*

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \Big|t_{2k,l}(x)\Big| &\le 0.6250 =: C_{even}, \\ \sum_{l\in\mathbb{Z}} \Big|t_{2k+1,l}(x)\Big| &\le 1.1430 =: C_{odd}.\end{aligned} \tag{7.22}$$

Proof. From (5.29) and fig. 8 we see that $\alpha_1(-c) = \alpha_0(c)$, and from Lemma 5.2 and fig. 7(b) we see that γ is an odd function. Now, using all these symmetries and the small values of the α's, it is not hard to observe that

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \left|t_{2k,l}(x)\right| &= \left(\frac{1}{4} - \Big(\alpha_2 + \frac{\alpha_2'}{\gamma'}(1-\gamma)\Big)(c_k)\right) + \left(\frac{1}{4} + \Big(\alpha_2 - \frac{\alpha_2'}{\gamma'}(1+\gamma)\Big)(c_k)\right) \\ &\le \sup_c \left(\frac{1}{2} - \frac{2\alpha_2'(c)}{\gamma'(c)}\right) = \frac{1}{2} - 2\inf_c\left(\frac{\alpha_2'(c)}{\gamma'(c)}\right) = 0.6250,\end{aligned}$$

and

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \left|t_{2k+1,l}(x)\right| &= \left(\frac{3}{32} + \Big(\alpha_1 + \frac{\alpha_1'}{\gamma'}(1-\gamma)\Big)(c_k)\right) + \left(\frac{3}{32} + \Big(\alpha_0 - \frac{\alpha_0'}{\gamma'}(1+\gamma)\Big)(c_{k+1})\right) \\ &\quad + \left(\frac{11}{16} + \Big(\alpha_1 - \frac{\alpha_1'}{\gamma'}(1+\gamma)\Big)(c_k) + \Big(\alpha_0 + \frac{\alpha_0'}{\gamma'}(1-\gamma)\Big)(c_{k+1})\right) \\ &\le \sup_{c,c'}\left(\frac{7}{8} + 2\big(\alpha_1 - \frac{\alpha_1'}{\gamma'}\gamma\big)(c) + 2\big(\alpha_0 - \frac{\alpha_0'}{\gamma'}\gamma\big)(c')\right) \\ &= \frac{7}{8} + 4\sup_c\left(\Big(\alpha_0 - \frac{\alpha_0'}{\gamma'}\gamma\Big)(c)\right) = 1.1430.\end{aligned}$$

The tedious analytical verification is left as an exercise. ☐

From this Lemma we see, first, that $\|T(x)\| = C_{odd} > 1$ and Theorem 6.2 doesn't hold for $n = 1$. On the other hand, since $C_{even}C_{odd} < 1$, it is obvious that (see (7.10))

$$\left|\big(T(x')T(x)w\big)_{2k}\right| \le C_{even}C_{odd}\|w\|, \qquad x', x, w \in l_\infty(\mathbb{Z}). \tag{7.23}$$

Furthermore, (7.23) holds for $4k + 1$, as well. To show this, we need the following fact.

Lemma 7.5 *For any $x, w \in l_\infty(\mathbb{Z})$ and $k \in \mathbb{Z}$*

$$|(T(x)w)_{2k} - (T(x)w)_{2k-1}| = |\triangle^1(T(x)w)_{2k-1}| \le 1.1965\|w\|. \tag{7.24}$$

Proof. First, let us observe some trivial properties of the entries of $T(x)$. For each $k \in \mathbb{Z}$, $t_{2k,k-1}, t_{2k,k}, t_{2k+1,k-1}$ are functions only on c_k, $t_{2k+1,k+1}$ is a function only on c_{k+1} and one can verify the following relations:

$$t_{2k,k-1}(-c_k) = t_{2k,k}(c_k); \qquad t_{2k+1,k-1}(-c_k) = t_{2k-1,k}(c_k); \qquad t_{2k+1,k+1}(-c_{k+1}) = t_{2k+3,k}(c_{k+1}).$$

Moreover, we can split $t_{2k+1,k}$ into two parts:

$$t^-_{2k+1,k}(c_k) := \frac{11}{32} + (\alpha_1 - \frac{\alpha_1'}{\gamma'}(1+\gamma))(c_k); \qquad t^+_{2k+1,k}(c_{k+1}) := \frac{11}{32} + (\alpha_0 + \frac{\alpha_0'}{\gamma'}(1-\gamma))(c_{k+1}).$$

Simple Matlab routines give

$$t_{2k,k-1}, t_{2k,k} \in [0.1684, 0.3636]; \qquad t_{2k+1,k-1}, t_{2k+1,k+1} \in [-0.2946, -0.02]; \qquad t_{2k+1,k} \in [0.4687, 1.0669].$$

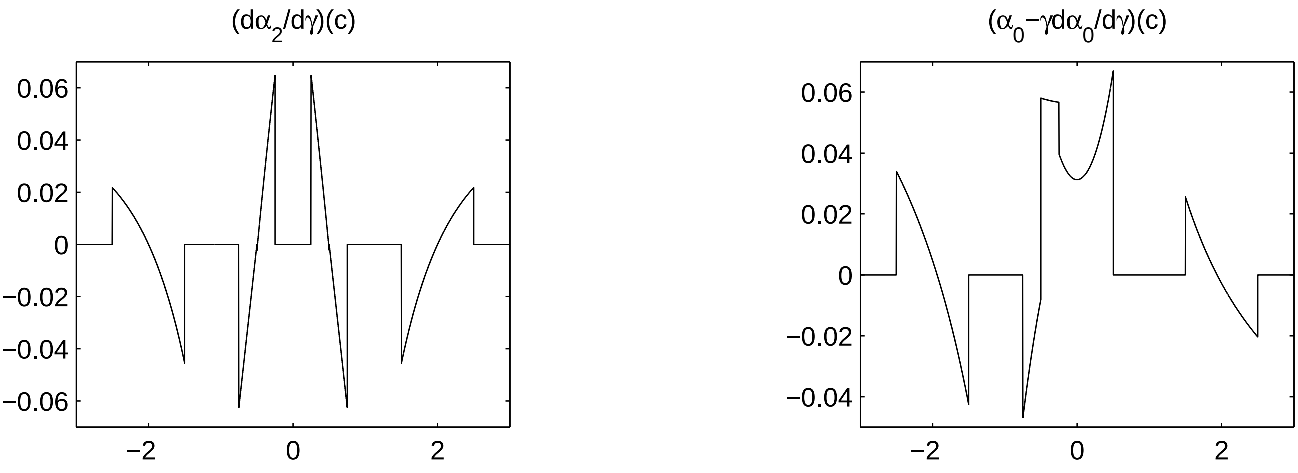

Figure 9: $\frac{\alpha_2'}{\gamma'}(c)$ and $\left(\alpha_0 - \frac{\alpha_0'}{\gamma'}\gamma\right)(c)$.

Now, let us denote by $w^1 := T(x)w$. Then (7.20) and (7.21), together with the above estimations, give rise to

$$\begin{aligned} |w^1_{2k} - w^1_{2k-1}| &= |-t_{2k-1,k-2}w_{k-2} + (t_{2k,k-1} - t_{2k-1,k-1})w_{k-1} + (t_{2k,k} - t_{2k-1,k})w_k| \\ &\le \big(-t_{2k-1,k-2} + t_{2k-1,k-1} - t_{2k,k-1} + t_{2k,k} - t_{2k-1,k}\big)\|w\| \\ &\le \big(\sup_{c_{k-1}}(t^-_{2k-1,k-1} - t_{2k-1,k-2}) + \sup_{c_k}(t^+_{2k-1,k-1} - t_{2k,k-1} + t_{2k,k} - t_{2k-1,k})\big)\|w\| \\ &\le (0.5715 + 0.6250)\|w\| = 1.1965\|w\|. \end{aligned} \tag{7.25}$$

□

Therefore

$$w^2_{4k+1} = (T(x')T(x)w)_{4k+1} = t_{4k+1,2k-1}w^1_{2k-1} + t_{4k+1,2k}w^1_{2k} + t_{4k+1,2k+1}w^1_{2k+1}. \tag{7.26}$$

Lemma 7.4 and the fact that $t_{4k+1,2k}$ is positive and $t_{4k+1,2k-1}, t_{4k+1,2k+1}$ are negative implies that $|w^2_{4k+1}|/\|w\| \le C_{even}C_{odd}$ whenever w^1_{2k} doesn't have the opposite sign of both w^1_{2k-1} and w^1_{2k+1} (the details are left to the reader). Now, without loss of generality, let us assume that w^1_{2k} is positive and w^1_{2k-1}, w^1_{2k+1} are negative. Let $w^1_{2k} = a\|w\|, a \in [0, 0.625]$. Applying Lemma 7.5 we derive

$$\begin{aligned} |w^2_{4k+1}| &\le \big(-t_{4k+1,2k-1}(1.1965 - a) + t_{4k+1,2k}a - t_{4k+1,2k+1}(1.1965 - a)\big)\|w\| \\ &\le \big((t_{4k+1,2k} - t_{4k+1,2k-1} - t_{4k+1,2k+1})a + (-t_{4k+1,2k-1} - t_{4k+1,2k+1})(1.1965 - 2a)\big)\|w\| \\ &\le \max\{C_{odd}C_{even}, (0.5892)(1.1965)\}\|w\| = C_{even}C_{odd}\|w\|. \end{aligned} \tag{7.27}$$

Unfortunately, such kind of contracting property does not hold for $4k+3$ even for the subdivision case, where computer simulations give that (see fig. 10) there exist $x, w \in l_\infty(\mathbb{Z})$ such that

$$|(T(S_{med,1}x)T(x)w)_{4k+3}| = 1.0669\|w\|. \tag{7.28}$$

However, there are strong numerical evidences that the "differential" approach works with $n = 3$. Indeed, due to Lemma 7.4, (7.23) and (7.27) we have:

Corollary 7.6 *Let $w^3 = T(x'')T(x')T(x)w$, where $x'', x', x, w \in l_\infty(\mathbb{Z})$ are arbitrary sequences. Then,*

$$\begin{aligned}
|w^3_{8k}|, |w^3_{8k+2}|, |w^3_{8k+4}|, |w^3_{8k+6}| &\le C_{even}\|w^2\| \le C_{even}C_{odd}^2\|w\| = 0.8165\|w\| \\
|w^3_{8k+1}|, |w^3_{8k+5}| &\le C_{even}C_{odd}\|w^1\| \le C_{even}C_{odd}^2\|w\| = 0.8165\|w\| \\
|w^3_{8k+3}| = \Big|\sum_{l=4k}^{4k+2} t_{8k+3,l}(x'')w^2_l\Big| &\le C_{odd}\max_{l=0,1,2}|w^2_{4k+l}| \le C_{even}C_{odd}^2\|w\| = 0.8165\|w\|.
\end{aligned}$$

Hence, the only "bad" case is $|w^3_{8k+7}|$.

$$\begin{array}{ccccccccccccc}
 & \bar{v}^0 \equiv \begin{bmatrix} v^0_{-1} \\ v^0_0 \\ v^0_1 \\ v^0_2 \end{bmatrix} & \xrightarrow{\bar{S}} & & \bar{v}^1 \equiv \begin{bmatrix} v^1_0 \\ v^1_1 \\ v^1_2 \\ v^1_3 \end{bmatrix} & \xrightarrow{\bar{S}} & & \bar{v}^2 \equiv \begin{bmatrix} v^2_2 \\ v^2_3 \\ v^2_4 \\ v^2_5 \end{bmatrix} & \xrightarrow{\bar{S}} & & \bar{v}^3 \equiv \begin{bmatrix} v^3_6 \\ v^3_7 \\ v^3_8 \\ v^3_9 \end{bmatrix} \\
\bar{c}^0 {\nearrow \atop \searrow} & \downarrow \triangle^1 & & \bar{c}^1 {\nearrow \atop \searrow} & \downarrow \triangle^1 & & \bar{c}^2 {\nearrow \atop \searrow} & \downarrow \triangle^1 & & \bar{c}^3 {\nearrow \atop \searrow} & \downarrow \triangle^1 \\
 & \bar{x}^0 \equiv \begin{bmatrix} x^0_{-1} \\ x^0_0 \\ x^0_1 \end{bmatrix} & \xrightarrow{\bar{S}_1} & & \bar{x}^1 \equiv \begin{bmatrix} x^1_0 \\ x^1_1 \\ x^1_2 \end{bmatrix} & \xrightarrow{\bar{S}_1} & & \bar{x}^2 \equiv \begin{bmatrix} x^2_2 \\ x^2_3 \\ x^2_4 \end{bmatrix} & \xrightarrow{\bar{S}_1} & & \bar{x}^3 \equiv \begin{bmatrix} x^3_6 \\ x^3_7 \\ x^3_8 \end{bmatrix}
\end{array}$$

Since S_{med} is local and shift-invariant, we reduce the problem of stability of the median-interpolating subdivision scheme to the study of 3×3 matrix products. This can be seen by the above diagram, where $\bar{S}$ are nonlinear maps on $\mathbb{R}^4$ (restrictions of S_{med} to the indicated coordinates) and $\bar{S}_1, \bar{T}$ are nonlinear maps on $\mathbb{R}^3$ (restrictions of $S_{med,1}$, respectively of T). Each of these maps depends on the underlying centers $\bar{c^j} \in \mathbb{R}^2$. In particular, $\bar{T} = \bar{T}(\bar{c^j})$, and we prefer to work with the more explicit expressions via $\bar{c^j}$.
In the multiresolution case, we use the same notation but we do not assume that $\bar{c^j}$ are produced by subdivision from one "seed" $\bar{v^0}$, i.e., we consider them unrelated.

The following result has only been verified by numerical methods (see fig. 11) and needs further scrutiny:

Proposition 7.7 *Let $\bar{v^0} = (v^0_{-1}, v^0_0, v^0_1, v^0_2)$ and $\bar{w} = (w_{-1}, w_0, w_1)$. Then*

$$\Big|\big(\bar{T}(\bar{c^2})\bar{T}(\bar{c^1})\bar{T}(\bar{c^0})\bar{w}\big)_{15}\Big| \le 0.81\|\bar{w}\|. \tag{7.29}$$

Thus, we can conclude:

Theorem 7.8 ***(Stability of median-interpolating subdivision scheme)*** *The dyadic median-interpolating subdivision scheme is stable!*

Proof. Corollary 7.6 and Proposition 7.7 give

$$\sup_{x \in l_\infty(\mathbb{Z})} \|T(S^2_{med,1}x)T(S_{med,1}x)T(x)\| \le 0.8165 < 1,$$

which, together with Proposition 6.21 and Lemma 7.3, implies Theorem 6.2. □

Remark 7.9 *In the multiresolution case, without additional restriction on the class of the admissible details, we do not expect stability, since there exists $v^0 \in l_\infty(\mathbb{Z})$ such that*

$$\rho(T(\triangle^1 v^0)) > 1.$$

8 Conclusions

In this thesis we presented a new stability theorem for univariate subdivision and multiresolution, showed that this theorem covers all the existing results for stability (WENO, PPH), and proved stability of a new subdivision scheme - the median-interpolating one. The joint spectral radius version of our theorem shows that for a large class of subdivision schemes the conditions of the theorem are "close to optimal", i.e., they are not only sufficient but also "almost" necessary (we use the word "almost" because we still do not know what happens when the joint spectral radius is exactly one).

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9 Appendix

Lemma 7.3:

Proof. We will use the notation from Section 5.3. Let us denote by v^1, resp. $\tilde{v}^1$ the refinement $S_{med}v$ of v, resp. $S_{med}\tilde{v}$ of $\tilde{v}$. Due to symmetry, locality, and shift-invariance, it is enough to show that

$$|v_0^1 - \tilde{v}_0^1| \le \max_{i=-1,0,1}\{|v_i - \tilde{v}_i|\} + K|\triangle^2(v-\tilde{v})_{-1}|. \tag{9.1}$$

Indeed, then $\|v^1 - \tilde{v}^1\| \le \|v-\tilde{v}\| + K\|\triangle^2(v-\tilde{v})\| \le \|v-\tilde{v}\| + 2K\|\triangle^1(v-\tilde{v})\|$.
Hence, we only need the finite subsequences $\bar{v} := [v_{-1}, v_0, v_1]$ and $\bar{\tilde{v}} := [\tilde{v}_{-1}, \tilde{v}_0, \tilde{v}_1]$. We will denote by $\triangle^2\bar{v} := \triangle^2 v_{-1}$, and by $\triangle^2\bar{\tilde{v}} := \triangle^2\tilde{v}_{-1}$. (5.28) gives rise to

$$\begin{aligned} v_0^1 - \tilde{v}_0^1 &= \frac{1}{32}\Big(5(v_{-1}-\tilde{v}_{-1}) + 30(v_0-\tilde{v}_0) - 3(v_1-\tilde{v}_1)\Big) + \tilde{\alpha}_0\triangle^2\bar{\tilde{v}} - \alpha_0\triangle^2\bar{v} \\ &= \frac{1}{4}(v_{-1}-\tilde{v}_{-1}) + \frac{3}{4}(v_0-\tilde{v}_0) + \frac{3}{32}\triangle^2(\bar{\tilde{v}}-\bar{v}) + \tilde{\alpha}_0\triangle^2\bar{\tilde{v}} - \alpha_0\triangle^2\bar{v}, \end{aligned} \tag{9.2}$$

where, for simplicity we have denoted $\alpha_{0,0}$ by α_0, resp. $\tilde{\alpha}_{0,0}$ by $\tilde{\alpha}_0$.
Using that $\alpha_0(c) = 0 = \alpha_0(\tilde{c})$ if $c, \tilde{c} \notin (-5/2,-3/2)\cup(-1/2,1/2)\cup(3/2,5/2)$ (see (5.29)), we derive from (9.2) that in this case (9.1) holds with $K = 3/32$.

From now on, w.l.o.g., we will assume that $\tilde{c} \in (-5/2,-3/2)\cup(-1/2,1/2)\cup(3/2,5/2)$.
Let us first deal with the degenerate case $\triangle^2 v = 0$ (i.e., $c = \infty$). (9.2), together with the obvious $|\tilde{\alpha}_0| \le 1/32$, implies that (9.1) holds with $K = 3/32 + 1/32 = 1/8$.

Thus from now on $c \in \mathbb{R}$, in addition to the restrictions on $\tilde{c}$. Using that S_{med} is affine-invariant and the canonical representation (5.25), we can reduce our investigations to

$$\begin{aligned} \bar{v} &= [1+\gamma, 0, 1-\gamma], & \gamma &= \gamma(c); \\ \bar{\tilde{v}} &= b + \big(1+\tfrac{d}{2}\big)[1+\tilde{\gamma}, 0, 1-\tilde{\gamma}], & \tilde{\gamma} &= \gamma(\tilde{c}). \end{aligned} \tag{9.3}$$

This scaling, together with (5.26), leads to

$$\triangle^2\bar{v} = 2, \qquad \triangle^2\bar{\tilde{v}} = 2+d, \qquad \triangle^2(\bar{\tilde{v}} - \tilde{v}) = d, \qquad \gamma \in \mathbb{R}, \qquad |\tilde{\gamma}| \le \frac{5}{2}.$$

Plain substitution in (9.2) gives

$$v_0^1 - \tilde{v}_0^1 = \underbrace{-b + \frac{1}{4}(\gamma - \tilde{\gamma}) + 2(\tilde{\alpha}_0 - \alpha_0)}_{A} + d\underbrace{\left(\tilde{\alpha}_0 + \frac{3}{32} - \frac{1+\tilde{\gamma}}{8}\right)}_{B}. \tag{9.4}$$

From $|\tilde{\alpha}_0| \le 1/32$ and $|\tilde{\gamma}| \le 5/2$, it follows that $-\frac{12}{32} \le B \le \frac{10}{32}$, and thus, in order to prove (9.1) it remains to show that $|A| \le \|\bar{v} - \bar{\tilde{v}}\| + K_1|d|$.

Lemma 9.1 $\|\gamma'(c)\| \ge \frac{11}{16}$ *and* $\|\alpha_0'(c)\| \le \frac{2}{15}$.

Proof. Lemma 5.2 and (5.29) imply that $\gamma, \alpha_0 \in W^1_\infty$. Moreover, $\gamma' \equiv 1,\ c \notin (-5/2, -3/2) \cup (-1/2, 1/2) \cup (3/2, 5/2)$, and $\alpha_0' \equiv 0,\ c \notin (-5/2, -3/2) \cup (-3/4, 1/2) \cup (3/2, 5/2)$.

Let us start with $\gamma(c)$.

$$\gamma'(c) = \frac{(32 + \epsilon_{-2}' - \epsilon_2')(32 + \epsilon_{-2} - 2\epsilon_0 + \epsilon_2) - (\epsilon_{-2}' - 2\epsilon_0' + \epsilon_2')(32c + \epsilon_{-2} - \epsilon_2)}{(32 + \epsilon_{-2} - 2\epsilon_0 + \epsilon_2)^2}.$$

Case I. $c \in [-1/2, 1/2]$. Then

$$\gamma'(c) = \frac{32(30 + 8c^2) - 16c32c}{(30 + 8c^2)^2} = \frac{32(30 - 8c^2)}{(30 + 8c^2)^2} \in \Big[\frac{7}{8}, \frac{16}{15}\Big].$$

We used that $\gamma'(c)$ is even and monotonically decreasing in $[0, 1/2]$.

Case II. $c \in [-5/2, -3/2] \cup [3/2, 5/2]$. Due to central symmetry (see fig 7b), the two cases are equivalent and w.l.o.g. $c \in [-5/2, -3/2]$.

$$\begin{aligned}\gamma'(c) &= \frac{(32 - 8(c+2))(33 - 4(c+2)^2) + 8(c+2)(32c + 1 - 4(c+2)^2)}{(33 - 4(c+2)^2)^2} \\ &= \frac{32(4(c+2)^2 - 24(c+2) + 33)}{(33 - 4(c+2)^2)^2} \in \Big[\gamma'\Big(-\frac{3}{2}\Big), \gamma'\Big(-\frac{5}{2}\Big)\Big] = \Big[\frac{11}{16}, \frac{23}{16}\Big].\end{aligned}$$

We used that $\gamma'(c)$ is monotonically decreasing in $[-5/2, -3/2]$.

For $\alpha_0(c)$ we proceed in the same way, but we have to consider more cases:

$$\alpha_0'(c) = \frac{(30\epsilon_0' + 5\epsilon_{-2}' - 3\epsilon_2' - 8\tilde{\epsilon}_{-1/2}')(32 + \epsilon_{-2} - 2\epsilon_0 + \epsilon_2) - (\epsilon_{-2}' - 2\epsilon_0' + \epsilon_2')(30\epsilon_0 + 5\epsilon_{-2} - 3\epsilon_2 - 8\tilde{\epsilon}_{-1/2})}{32(32 + \epsilon_{-2} - 2\epsilon_0 + \epsilon_2)^2}.$$

Case I. $c \in [-5/2, -3/2] \cup [-1/4, 1/2] \cup [3/2, 5/2]$. Using (5.29) and that ϵ_{-2} and ϵ_2 are just shifts of ϵ_0, we derive that (after shifting the argument, if necessary)

$$\alpha_0 = \frac{C_1 \epsilon_0}{32(32 + C_2\epsilon_0)} \Longrightarrow \alpha_0' = C_1 \frac{\epsilon_0'}{(32 + C_2\epsilon_0)^2},$$

where C_1 equals to $5, 30$, resp. -3 and C_2 equals to $-1, 2$, resp. -1, when c lies in $[-5/2, -3/2]$, $[-1/4, 1/2]$, resp. $[3/2, 5/2]$. In all the three cases, direct computations show that $\frac{\epsilon_0'}{(32+C_2\epsilon_0)^2}$ is monotonically decreasing, and $\big|\frac{\epsilon_0'}{(32+C_2\epsilon_0)^2}\big| \le \frac{1}{16^2}$. Thus

$$|\alpha_0'(c)| \le \begin{cases} 5/16^2, & c \in [-5/2, -3/2] \\ 30/16^2, & c \in [-1/4, 1/2] \\ 3/16^2, & c \in [3/2, 5/2] \end{cases}.$$

Case II. $c \in [-3/4, -1/2]$. Then

$$\alpha_0'(c) = \frac{1 + 2c}{8} \in \Big[-\frac{1}{16}, 0\Big].$$

Case III. $c \in [-1/2, -1/4]$. Then

$$\alpha_0'(c) = \frac{-8c^2 - 3c + 30}{(15 + 4c^2)^2} \in \Big[\alpha_0'\Big(-\frac{1}{2}\Big), \alpha_0'\Big(-\frac{1}{4}\Big)\Big] \subset \Big[0, \frac{2}{15}\Big].$$

We used that $\alpha_0'(c)$ is positive and monotonically decreasing, as well as that

$$\alpha_0'\big(-\frac{1}{4}\big) = \frac{2\big(15 + \frac{1}{8}\big)}{\big(15 + \frac{1}{4}\big)\big(15 + \frac{1}{4}\big)} < \frac{2}{15}.$$

□

Now using Lemma 9.1, and that both $\alpha_0(c)$ and $c(\gamma)$ are continuous and piecewise differentiable, we derive

$$|2(\tilde{\alpha}_0 - \alpha_0)| \le \|\alpha_0'\||c - \tilde{c}| \le \|\alpha_0'\|\|c'\||\gamma - \tilde{\gamma}| \le \frac{4}{15}\frac{16}{11}|\gamma - \tilde{\gamma}| \le \frac{2}{5}|\gamma - \tilde{\gamma}|, \tag{9.5}$$

which gives rise to

$$|A| \le \max\{|-b + (\gamma - \tilde{\gamma})|, |-b - (\gamma - \tilde{\gamma})|\}. \tag{9.6}$$

On the other hand (9.3) implies

$$\begin{aligned} \|\bar{v} - \bar{\tilde{v}}\| &= \Big\|[-b + (\gamma - \tilde{\gamma}), 0, -b - (\gamma - \tilde{\gamma})] - \frac{d}{2}[1 + \tilde{\gamma}, 0, 1 - \tilde{\gamma}]\Big\| \\ &\ge \max\{|-b + (\gamma - \tilde{\gamma})|, |-b - (\gamma - \tilde{\gamma})|\} - \frac{|d|}{2}\max\{|1 + \tilde{\gamma}|, |1 - \tilde{\gamma}|\}. \end{aligned} \tag{9.7}$$

Finally, combining (9.6) and (9.7), and recalling that $|\tilde{\gamma}| \le 5/2$, we conclude that

$$|A| \le \|\bar{v} - \bar{\tilde{v}}\| + \frac{7}{4}|d|. \tag{9.8}$$

□

Remark 9.2 *In order to simplify the exposition, as well as to decrease the number of cases, we used very rough inequalities, and thus the actual value of K is much smaller than the above estimation! However, the exact value of K is of no interest for proving stability of S_{med}.*

Remark 9.3 *The same approach works also for the triadic median-interpolating subdivision scheme. There (being more precise in the estimations) we have proven that*

$$\|S_{med,3}v - S_{med,3}\tilde{v} \le \|v - \tilde{v}\| + \frac{1}{3}\|\triangle^2(v - \tilde{v})\|.$$

Again, $1/3$ is not the exact lower bound for the constant K, but numerical simulations show that $K > 0.29$.

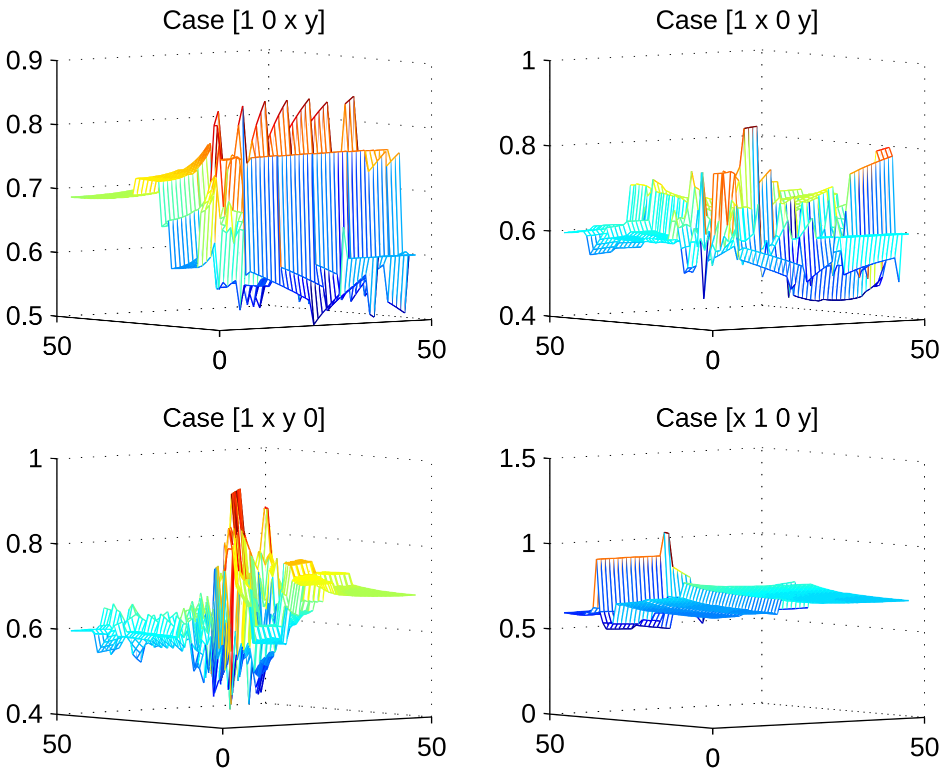

Figure 10: $\sup_{\|w\|=1} |T(S_{med,1}x)T(x)w_7|$ as a function of the centers c_0 and c_1 of v, where $x = \triangle^1 v$.

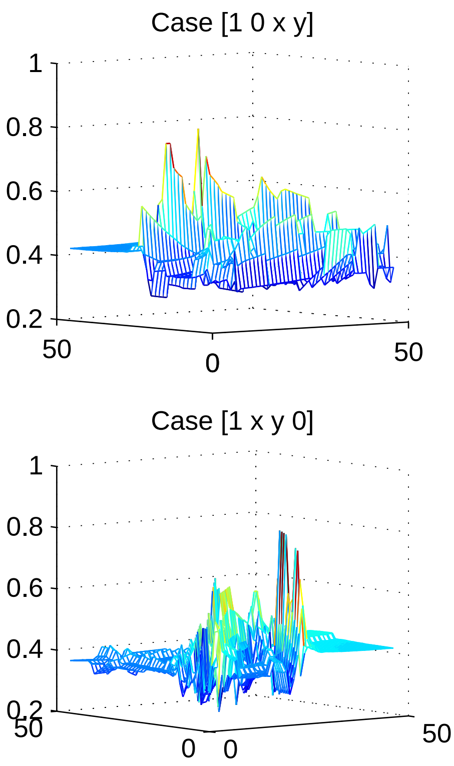

Figure 11: $\sup_{\|w\|=1} |T(S^2_{med,1}x)T(S_{med,1}x)T(x)w_{15}|$ as a function of the centers c_0 and c_1 of v, where $x = \triangle^1 v$.