A third order scheme for Hamilton-Jacobi equations on triangular grids

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Presentation outline

- Brief overview of numerical methods for Hamilton-Jacobi equations
- A conforming, piecewise quadratic scheme on triangular meshes, with local evolution for Hamilton-Jacobi equations
- Numerical examples
- Conclusions
The Hamilton-Jacobi Equation

We are interested in computing numerical solutions to the Cauchy problem for the Hamilton-Jacobi equation:

\[ u_t(x, t) + H(x, \nabla u) = 0 \quad \text{for } \forall (x, t) \in \mathbb{R}^n \times [0, T] \]  
\[ u(0, x) = \tilde{u}(x) \quad \text{for } \forall x \in \mathbb{R}^n \]  

Applications
- Plasma processes in semiconductor industry
- Image processing
- Optimal Control
- Problems with evolving interfaces: crack growth, multiphase flow, etc.
Theoretical Background

- There exist infinitely many Lipschitz-continuous solutions to (1).
- Uniqueness is obtained by considering viscosity solutions:

\[
  u^\varepsilon_t + H(x, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon
\]

(2)

The (uniform) limit \( u^\varepsilon \to u \) when \( \varepsilon \to 0, \varepsilon > 0 \), if it exists, is called a viscosity solution of (1).

- Assume that \( H \) satisfies the assumptions:
  1. \( |H(x, p) - H(y, p)| \leq C |x - y| (1 + |p|) \)
  2. \( |H(x, p) - H(x, q)| \leq C |p - q| \)

Then the Hamilton-Jacobi equation (1) admits a unique viscosity solution.
Semi-discrete methods in 1D

- At time $t = t_n$, find an interior to each cell, where the solution will remain smooth for the entire duration $dt$ of the time step.

- Use the smooth interior solution to reconstruct a value for the solution at the mesh nodes.

- Take the limit $dt \to 0$ and derive an ODE for the cell nodes. For example (Bryson, et al):

\[
\frac{du_i}{dt}(t_n) = - \frac{a_i^- H(u_x^+) + a_i^+ H(u_x^-)}{a_i^+ + a_i^-} \\
+ a_i^- a_i^+ \left[ \frac{u_x^+ - u_x^-}{a_i^+ + a_i^-} - \minmod\left( \frac{u_x^+ - \tilde{u}_x}{a_i^+ + a_i^-}, \frac{\tilde{u}_x - u_x^-}{a_i^+ + a_i^-} \right) \right]
\]  

- The ODE is defined only for the mesh nodes, but not the midpoints!
Semi-discrete methods in 1D (cont.)

- Given the known piecewise quadratic approximation of the solution at \( t = t_n \), make one time step of the ODE to obtain values at the mesh nodes, i.e. \( u(x_i, t_{n+1}) \).
- Based on the computed \( u(x_i, t_{n+1}) \), reconstruct the values at the midpoints \( u(x_{i+\frac{1}{2}}, t_{n+1}) \) by minimizing convexity, i.e., minmod limiter scheme:
Numerical methods in 2D

- ENO (Essentially Non-oscillatory Methods), WENO (Weighted ENO) (e.g. Osher, Sethian, Shu).
- Semi-discrete methods on structured grids with line reconstructions (e.g. Bryson, Kurganov, Levy, Petrova)
Current Method: Basic Idea

- Use a piecewise quadratic, conforming approximation of $u(\cdot, t)$ on triangles, for any given time $t$.

- Every time-step consists of the following substeps:
  - Local evolution of the solution in the interior of each triangle
  - Reconstruction of the solution on the original grid (vertices and midpoints) from the interior quadratic polynomials
Local Evolution

- For each element $e$, select an interior triangle, homothetic to $e$, such that the solution remains smooth for the duration of the time step.

- Let $u_{e}^{int}$ be the restriction of $u(\cdot, t_n)$ over this interior triangle.

- Evolve each interior restriction $u_{e}^{int}$ by a suitable integrator, that is, solve numerically

  $$ \frac{d u_{e}^{int}}{dt} = -H(x, \nabla u_{e}^{int}) $$

  by a second order method to obtain $u_{e}^{int}(\cdot, t_{n+1})$.

- At the end, one has an piecewise quadratic, discontinuous approximation to the solution at $t = t_{n+1}$.
Reconstruction I: node based

- For each triangle $e$, construct the interior and exterior interpolants $u_e^{int}$ and $u_e^{ext}$, respectively.
- Choose the interpolant which has lower convexity.
- For each node $v$ (vertex or midpoint), consider all upwind triangles $\{e^i_v\}_{i \in U_v}$ and let $u_v$ be the one with lowest convexity.
- The nodal value at $v$ is assigned the value of the upwind interpolant with lowest convexity, that is,
  \[ u(v, t_{n+1}) = u_v(v). \]

When the above procedure is repeated for all vertices and midpoints, one has a continuous, piecewise quadratic approximation of $u$ at time step $t = t_{n+1}$. 
For each triangle $e$, consider the values of the interior interpolant $u_{e}^{int}$ as data.

Use the values inside $e$ and its neighbors to generate quadratic functions which interpolate six of the data points.

Choose the approximant inside $e$ which has lowest convexity from the admissible set of quadratic functions.

For each node $v$ (vertex or midpoint), the value assigned is the average of all approximants.

When the above procedure is repeated for all nodes, one has a unique continuous, piecewise quadratic interpolant of the data which is our approximation of $u$ at time step $t = t_{n+1}$.
Linear transport \((H(u_x, u_y) = u_x + u_y), \ h = 0.08, \ dt = 0.01\)

Initial Condition

\[ T=0 \] \[ T=0.25 \] \[ T=0.5 \] \[ T=0.75 \] \[ T=1 \]
Numerical Examples: Linear Transport

Linear transport \((H(u_x, u_y) = u_x + u_y), h = 0.2, dt = 0.01\)
Numerical Examples: Quadratic Hamiltonian

Nonlinear and convex Hamiltonian \( H(u_x, u_y) = u_x^2 + u_y^2 \), \( h \approx 0.2 \), \( dt = 0.0025 \), Smooth initial data.

Table 1: Relative \( L_1 \) error

<table>
<thead>
<tr>
<th>( T )</th>
<th>( h, dt )</th>
<th>( h/2, dt/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rec I</td>
<td>Rec II</td>
</tr>
<tr>
<td>0.1</td>
<td>0.026</td>
<td>0.021</td>
</tr>
<tr>
<td>0.15</td>
<td>0.034</td>
<td>0.024</td>
</tr>
<tr>
<td>0.2</td>
<td>0.040</td>
<td>0.028</td>
</tr>
<tr>
<td>0.3</td>
<td>0.054</td>
<td>0.038</td>
</tr>
<tr>
<td>0.4</td>
<td>0.071</td>
<td>0.048</td>
</tr>
</tbody>
</table>
Nonlinear and convex Hamiltonian \( H(u_x, u_y) = u_x^2 + u_y^2 \), \( h \approx 0.2 \), \( dt = 0.0025 \), Non-smooth initial data, Reconstruction I.
Numerical Examples: Quadratic Hamiltonian

Nonlinear and convex Hamiltonian \( H(u_x, u_y) = u_x^2 + u_y^2 \), \( h \approx 0.2, \ dt = 0.0025 \), Non-smooth initial data, Reconstruction II.
Numerical Examples: 2D Burgers

Nonlinear and convex Hamiltonian \( H(u_x, u_y) = \frac{1}{2}(u_x + u_y + 1)^2 \), 30 \times 30 grid, \( \Omega = [-2, 2]^2 \), \( dt = 0.0025 \), Reconstruction II.

Initial condition: \( u(x) = -\frac{1}{2} \cos(\pi(x + y)) \)

Solution at \( t = 1.5/\pi \).
Conclusions

- The proposed fully discrete method solves successfully linear and convex Hamilton-Jacobi equations on unstructured triangular grids.
- The method is exact for quadratic polynomials.
- Numerical experiments suggest that the reconstruction used is successful at limiting the convexity of the solution.
- A further analysis of the algorithm is needed to understand:
  - Stability of solution with respect to mesh parameters
  - Behavior of algorithm for non-convex Hamiltonians