A third order scheme for Hamilton-Jacobi equations on triangular grids

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Presentation outline

- Brief overview of numerical methods for Hamilton-Jacobi equations
- A conforming, piecewise quadratic scheme on triangular meshes, with local evolution for Hamilton-Jacobi equations
- Numerical examples
- Conclusions

The Hamilton-Jacobi Equation

- We are interested in computing numerical solutions to the Cauchy problem for the Hamilton-Jacobi equation:

$$
\begin{aligned}
u_t(\mathbf{x},t) + H(\mathbf{x},\nabla u) &= 0 \qquad \text{for } \forall(\mathbf{x},t) \in \mathbb{R}^n \times [0,T] \qquad (1)\\
u(0,\mathbf{x}) &= \tilde{u}(\mathbf{x}) \qquad \text{for } \forall \mathbf{x} \in \mathbb{R}^n
\end{aligned}
$$

- Applications
  - Plasma processes in semiconductor industry
  - Image processing
  - Optimal Control
  - Problems with evolving interfaces: crack growth, multiphase flow, etc.

Theoretical Background

- There exist infinitely many Lipschitz-continuous solutions to (1).
- Uniqueness is obtained by considering viscosity solutions:

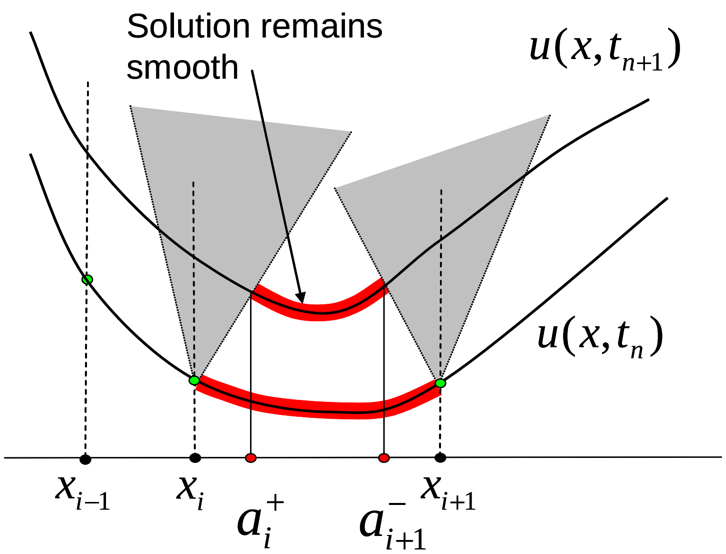
$$u_t^\varepsilon + H(\mathbf{x}, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon \quad (2)$$

  The (uniform) limit $u^\varepsilon \to u$ when $\varepsilon \to 0$, $\varepsilon > 0$, if it exists, is called a viscosity solution of (1).
- Assume that H satisfies the assumptions:
  1. $|H(\mathbf{x}, \mathbf{p}) - H(\mathbf{y}, \mathbf{p})| \leq C\, |\mathbf{x} - \mathbf{y}|\, (1 + |\mathbf{p}|)$
  2. $|H(\mathbf{x}, \mathbf{p}) - H(\mathbf{x}, \mathbf{q})| \leq C\, |\mathbf{p} - \mathbf{q}|$

  Then the Hamilton-Jacobi equation (1) admits a unique viscosity solution.

Semi-discrete methods in 1D

- At time $t = t_n$, find an interior to each cell, where the solution will remain smooth for the entire duration dt of the time step.
- Use the smooth interior solution to reconstruct a value for the solution at the mesh nodes.



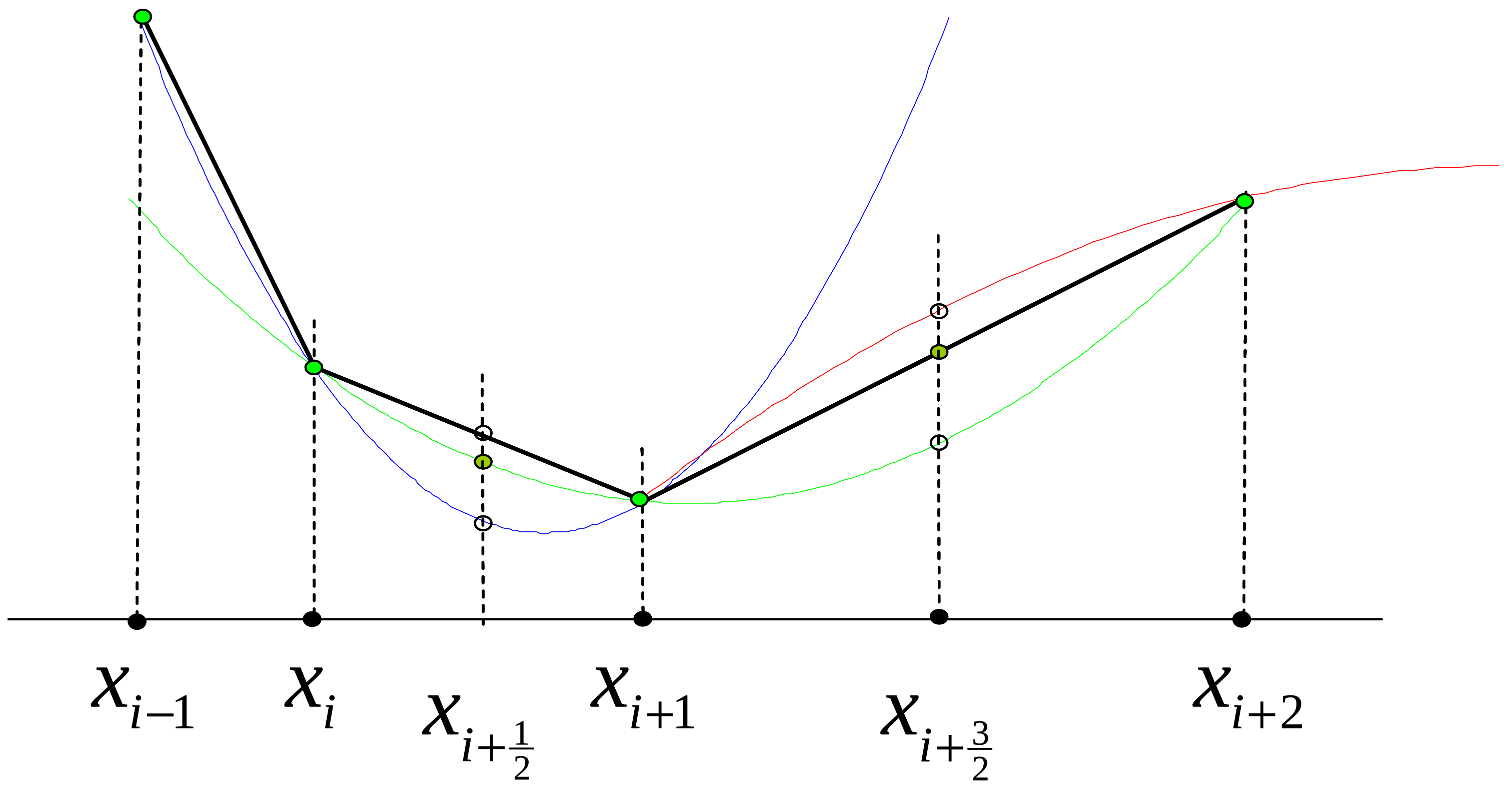
- Take the limit $dt \to 0$ and derive an ODE for the cell nodes. For example (Bryson, et al):

$$\frac{du_i}{dt}(t_n) = -\frac{a_i^- H(u_x^+) + a_i^+ H(u_x^-)}{a_i^+ + a_i^-} + a_i^- a_i^+ \left[\frac{u_x^+ - u_x^-}{a_i^+ + a_i^-} - \operatorname{minmod}\left(\frac{u_x^+ - \tilde{u}_x}{a_i^+ + a_i^-}, \frac{\tilde{u}_x - u_x^-}{a_i^+ + a_i^-} \right) \right] \quad (3)$$

- The ODE is defined only for the mesh nodes, but not the midpoints!

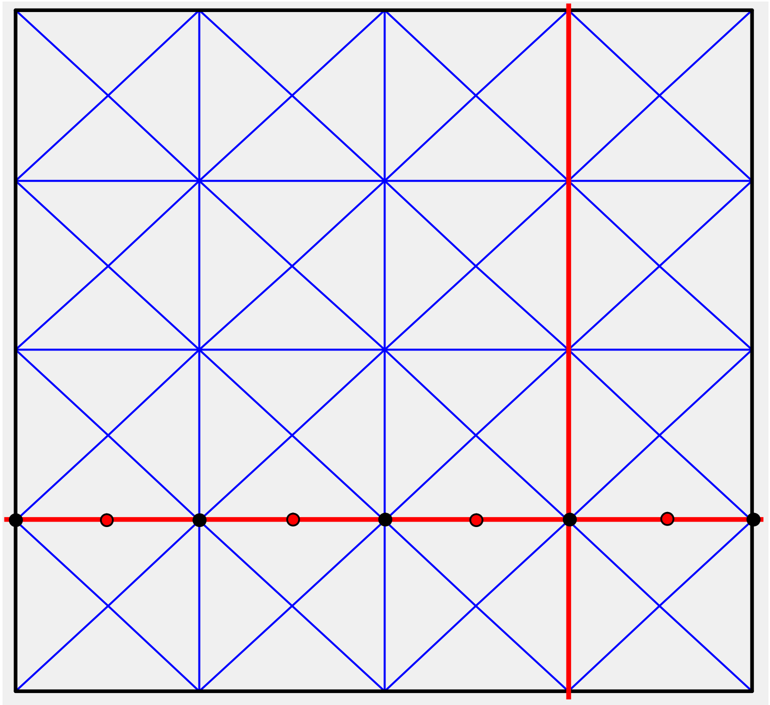
Semi-discrete methods in 1D (cont.)

- Given the known piecewise quadratic approximation of the solution at $t = t_n$, make one time step of the ODE to obtain values at the mesh nodes, i.e. $u(x_i, t_{n+1})$.
- Based on the computed $u(x_i, t_{n+1})$, reconstruct the values at the midpoints $u(x_{i+\frac{1}{2}}, t_{n+1})$ by minimizing convexity, i.e., minmod limiter scheme:



Numerical methods in 2D

- ENO (Essentially Non-oscillatory Methods), WENO (Weighted ENO) (e.g. Osher, Sethian, Shu).
- Semi-discrete methods on structured grids with line reconstructions (e.g. Bryson, Kurganov, Levy, Petrova)

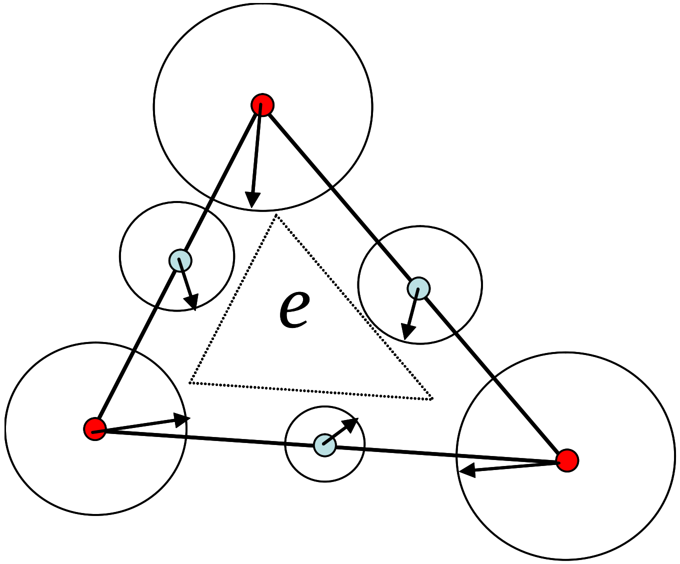


Current Method: Basic Idea

- Use a piecewise quadratic, conforming approximation of $u(:, t)$ on triangles, for any given time t.
- Every time-step consists of the following substeps:
  - Local evolution of the the solution in the interior of each triangle
  - Reconstruction of the solution on the original grid (vertices and midpoints) from the interior quadratic polynomials

Local Evolution

- For each element e, select an interior triangle, homothetic to e, such that the solution remains smooth for the duration of the time step.
- Let u_e^{int} be the restriction of $u(\cdot, t_n)$ over this interior triangle.



- Evolve each interior restriction u_e^{int} by a suitable integrator, that is, solve numerically

$$\frac{du_e^{int}}{dt} = -H(\mathbf{x}, \nabla u_e^{int}) \tag{4}$$

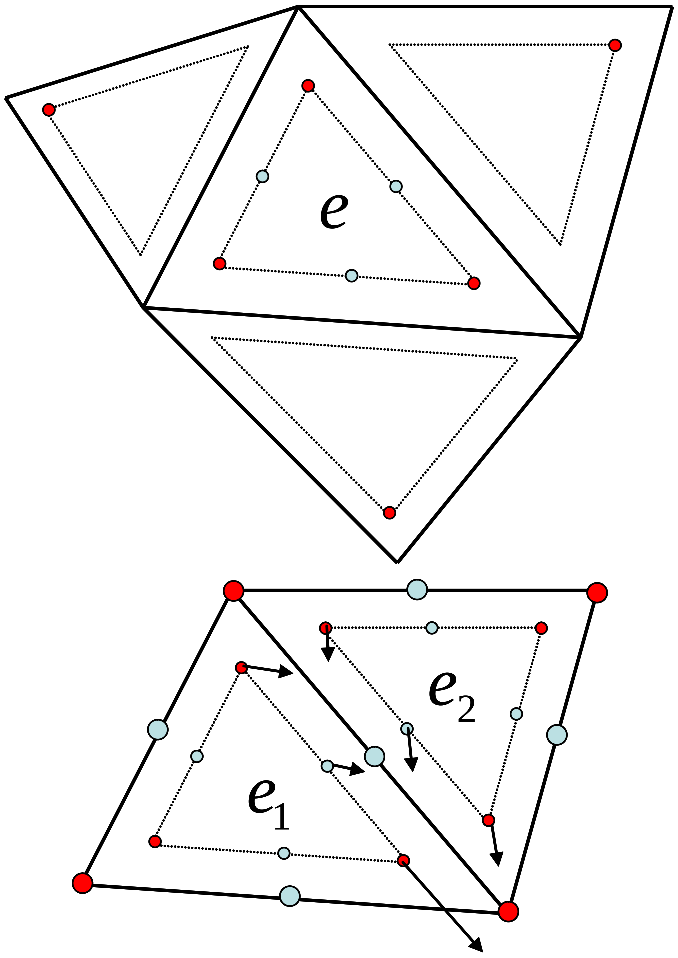
by a second order method to obtain $u_e^{int}(\cdot, t_{n+1})$.

- At the end, one has an piecewise quadratic, discontinuous approximation to the solution at $t = t_{n+1}$

Reconstruction I: node based

- For each triangle e, construct the interior and exterior interpolants u_e^{int} and u_e^{ext}, respectively.
- Choose the interpolant which has lower convexity
- For each node $\mathbf{v}$ (vertex or midpoint), consider all upwind triangles $\left\{e_{\mathbf{v}}^i\right\}_{i\in U_{\mathbf{v}}}$ and let $u_{\mathbf{v}}$ be the one with lowest convexity.
- The nodal value at $\mathbf{v}$ is assigned the value of the upwind interpolant with lowest convexity, that is,

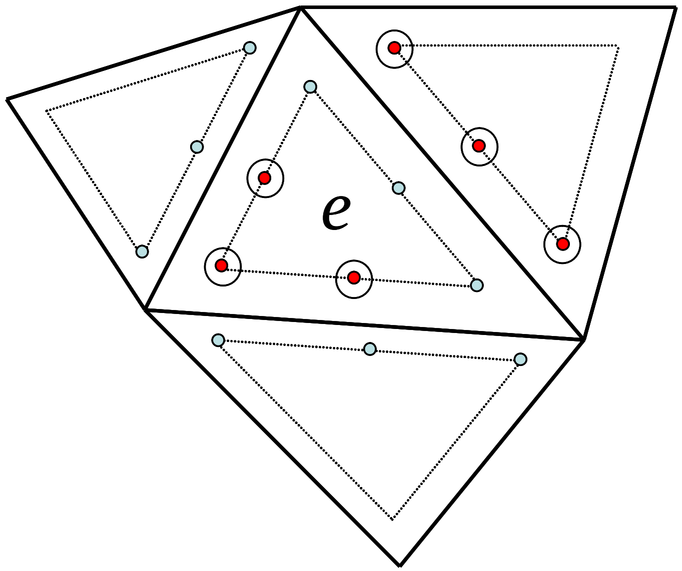
$$u(\mathbf{v}, t_{n+1}) = u_{\mathbf{v}}(\mathbf{v}).$$



When the above procedure is repeated for all vertices and midpoints, one has a continuous, piecewise quadratic approximation of u at time step $t = t_{n+1}$

Reconstruction II: triangle convexity

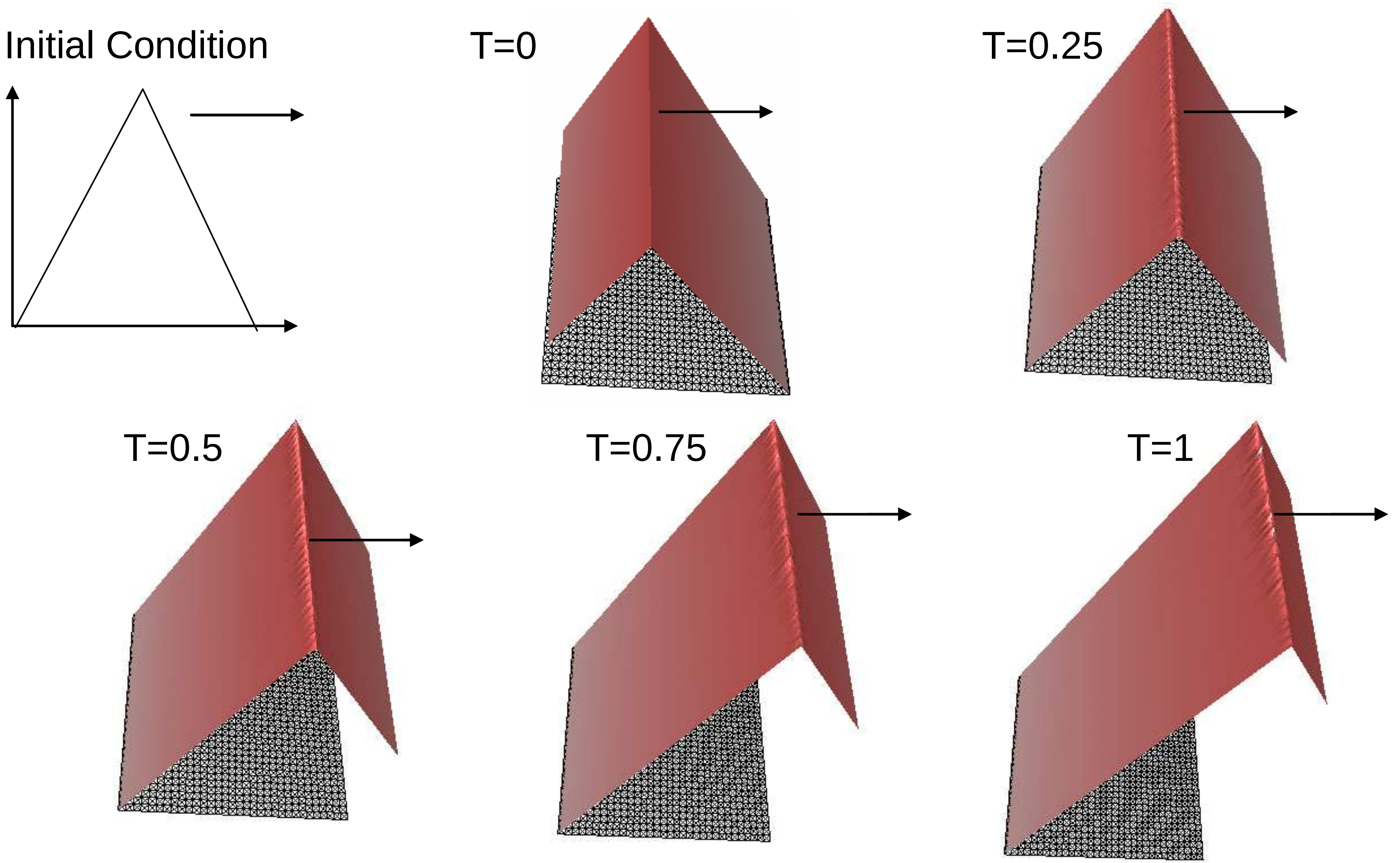
- For each triangle e, consider the values of the interior interpolant u_e^{int} as data.
- Use the values inside e and its neighbors to generate quadratic functions which interpolate six of the data points.
- Choose the approximant inside e which has lowest convexity from the admissible set of quadratic functions
- For each node $\mathbf{v}$ (vertex or midpoint), the value assigned is the average of all approximants



When the above procedure is repeated for all nodes, one has a unique continuous, piecewise quadratic interpolant of the data which is our approximation of u at time step $t = t_{n+1}$

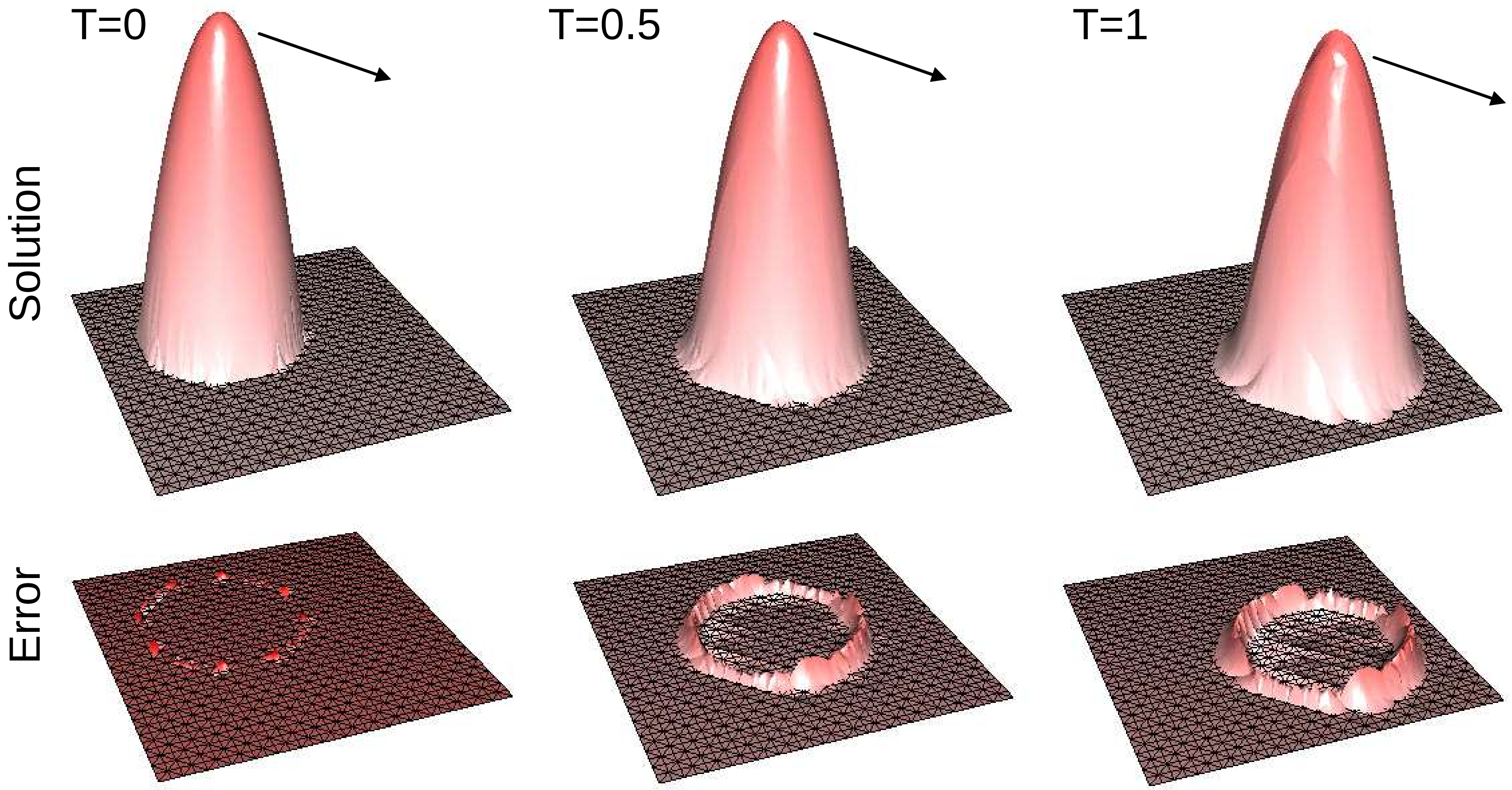
Numerical Examples: Linear Transport

Linear transport ($H(u_x, u_y) = u_x + u_y$), $h = 0.08$, $dt = 0.01$



Numerical Examples: Linear Transport

Linear transport ($H(u_x, u_y) = u_x + u_y$), $h = 0.2$, $dt = 0.01$



Numerical Examples: Quadratic Hamiltonian

Nonlinear and convex Hamiltonian ($H(u_x, u_y) = u_x^2 + u_y^2$), $h \approx 0.2$, $dt = 0.0025$, Smooth initial data.

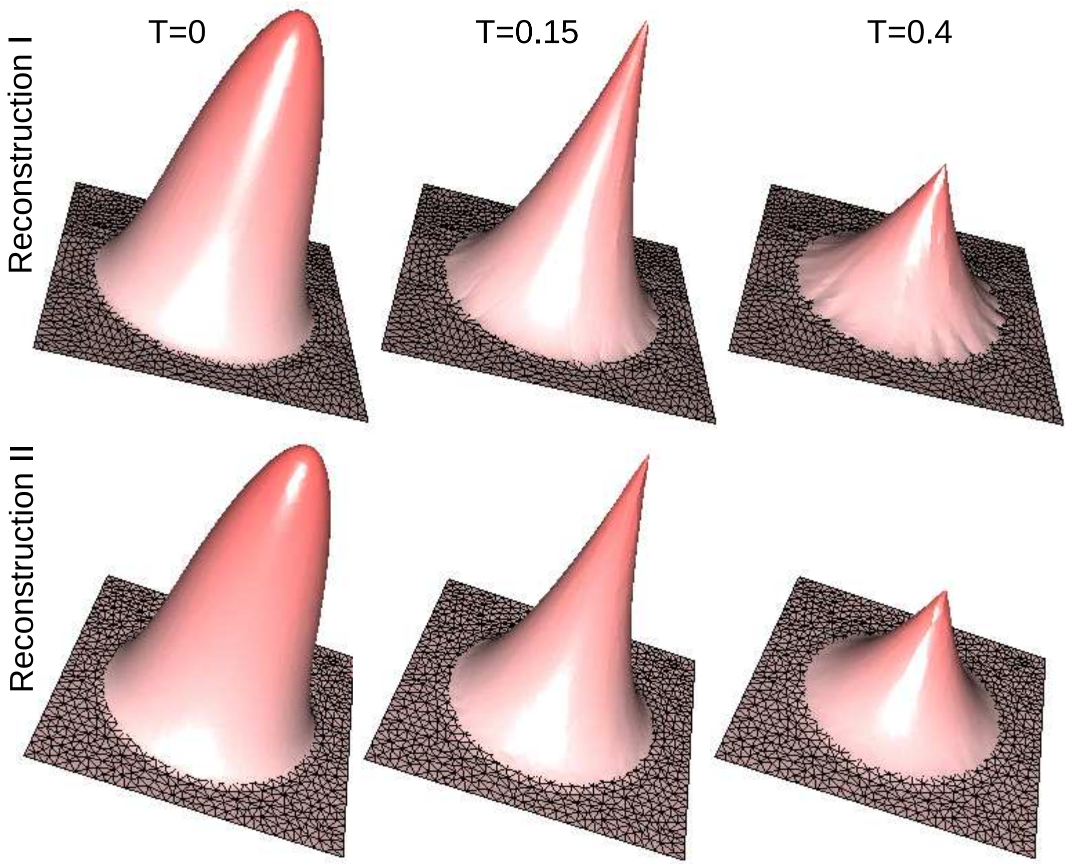
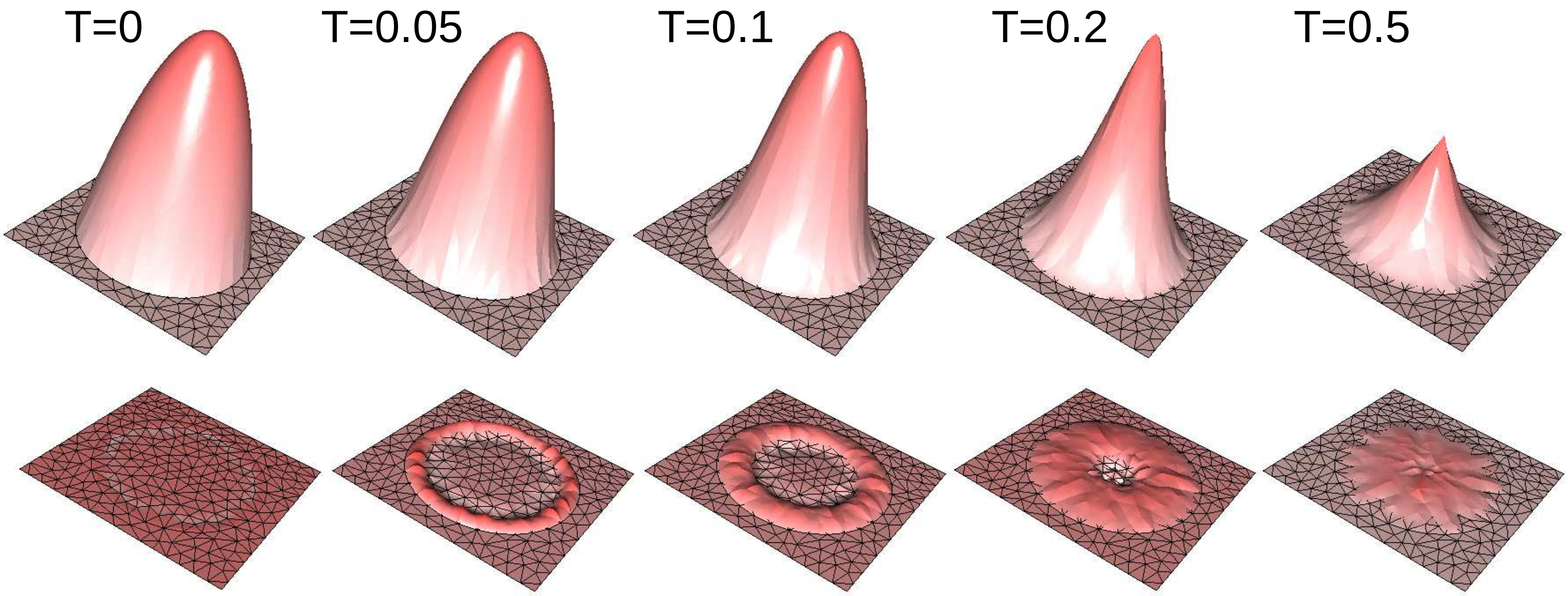


Table 1: Relative L_1 error

T	h, dt		$h/2$, $dt/2$	
	Rec I	Rec II	Rec I	Rec II
0.1	0.026	0.021	0.0064	0.004
0.15	0.034	0.024	0.0078	0.0046
0.2	0.040	0.028	0.0099	0.0058
0.3	0.054	0.038	0.014	0.0079
0.4	0.071	0.048	0.019	0.011

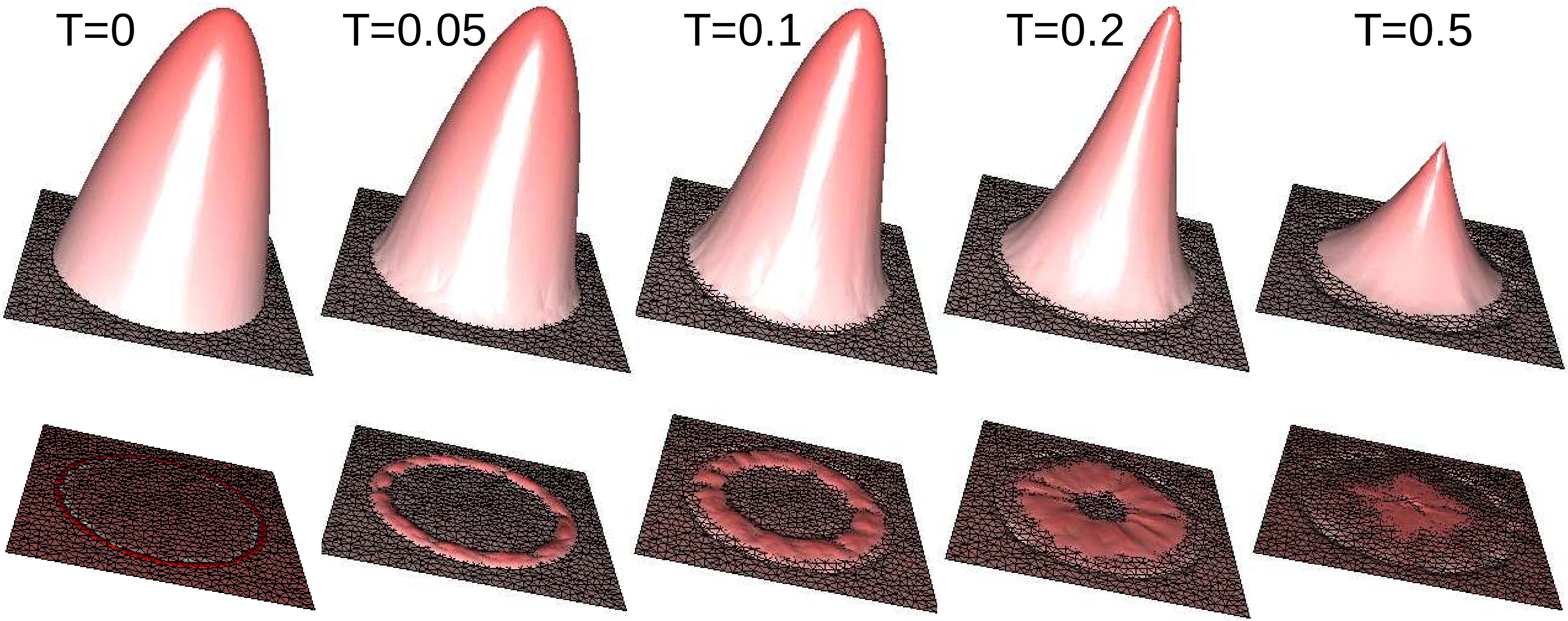
Numerical Examples: Quadratic Hamiltonian

Nonlinear and convex Hamiltonian ($H(u_x, u_y) = u_x^2 + u_y^2$), $h \approx 0.2$, $dt = 0.0025$, Non-smooth initial data, Reconstruction I.



Numerical Examples: Quadratic Hamiltonian

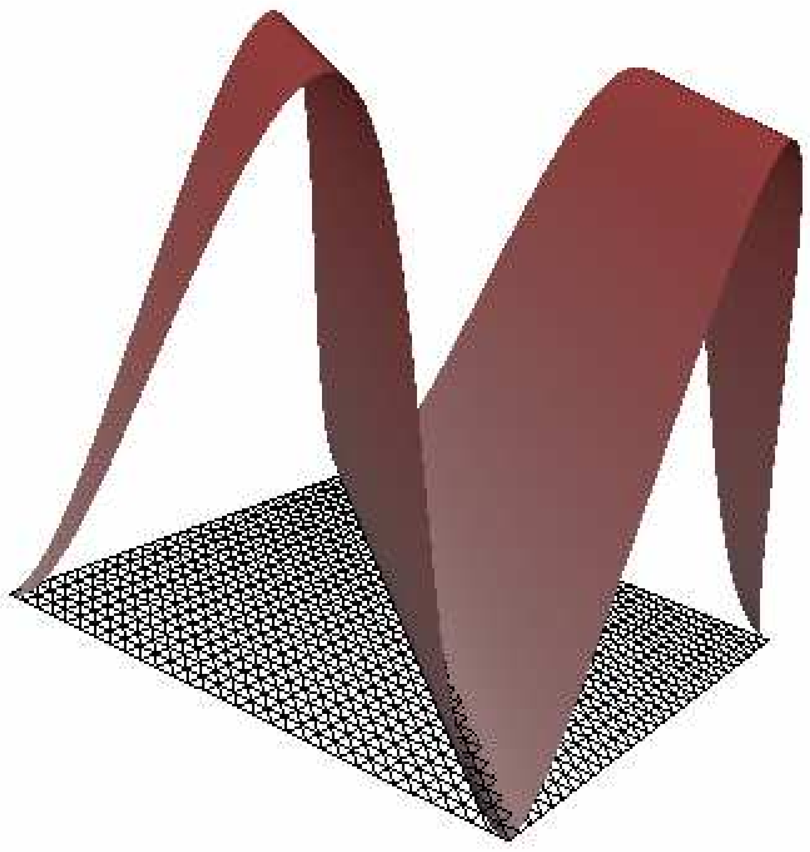
Nonlinear and convex Hamiltonian ($H(u_x, u_y) = u_x^2 + u_y^2$), $h \approx 0.2$, $dt = 0.0025$, Non-smooth initial data, Reconstruction II.



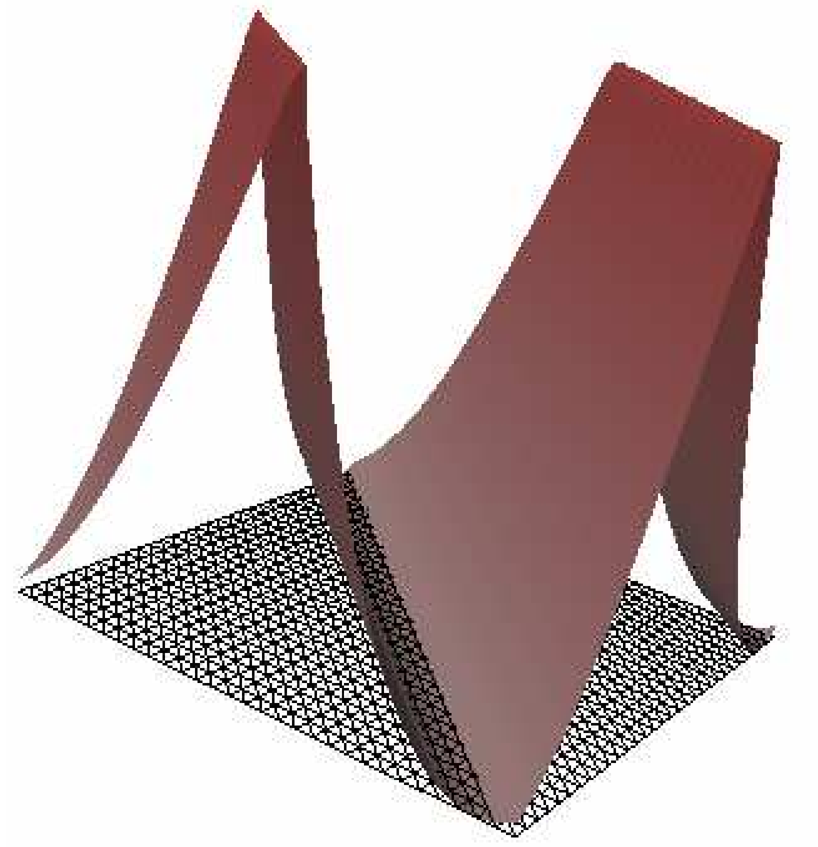
Numerical Examples: 2D Burgers

Nonlinear and convex Hamiltonian ($H(u_x, u_y) = \frac{1}{2}(u_x + u_y + 1)^2$), 30×30 grid, $\Omega = [-2, 2]^2$, $dt = 0.0025$, Reconstruction II.

Initial condition: $u(\mathbf{x}) = -\frac{1}{2}\cos(\pi(x + y))$



Solution at $t = 1.5/\pi$.



Conclusions

- The proposed fully discrete method solves successfully linear and convex Hamilton-Jacobi equations on unstructured triangular grids
- The method is exact for quadratic polynomials.
- Numerical experiments suggest that the reconstruction used is successful at limiting the convexity of the solution.
- A further analysis of the algorithm is needed to understand:
  - Stability of solution with respect to mesh parameters
  - Behavior of algorithm for non-convex Hamiltonians