MIC(0) Preconditioning of Conforming and Non-conforming FEM Problems on Unstructured Grids*

Nikola Kosturski and Svetozar Margenov

Institute for Parallel Processing, Bulgarian Academy of Sciences

Abstract In this study, the topics of grid generation and FEM applications are studied together following their natural synergy. We consider the following three grid generators: Triangle, NETGEN and Gmsh. The quantitative analysis is based on the number of elements/nodes needed to obtain a triangulation of a given domain, satisfying a certain minimal angle condition. After that, the performance of the MIC(0) preconditioned conjugate gradient (PCG) solver is analyzed for both conforming and non-conforming linear FEM problems. If positive off-diagonal entries appear in the corresponding matrix, a diagonal compensation is applied to get an auxiliary M-matrix allowing a stable MIC(0) factorization. Uniform estimates of the related relative condition numbers are derived.
The presented numerical experiments for elliptic and parabolic problems well illustrate the similar PCG convergence rate of the MIC(0) preconditioner for both, structured and unstructured grids. The comparative analysis of the performance for the cases of conforming (Courant) and non-conforming (Crouzeix-Raviart) finite elements is among the contributions of this paper.

Keywords: conforming and non-conforming FEM, unstructured grids, preconditioning, MIC(0) factorization

1 Introduction

Mesh generation techniques are now widely employed in various scientific and engineering fields that make use of physical models based on partial differential equations. While there are a lot of works devoted to finite element methods (FEM) and their applications, it appears that the issues of meshing technologies in this context are less investigated. Thus, in the best cases, this aspect is briefly mentioned as a technical point that is possibly non-trivial. In this study, the topics of grid generation and FEM applications are studied together following their natural synergy.

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Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$. The following elliptic

$$\begin{aligned} -\nabla \cdot (a(\mathbf{x})\nabla u(\mathbf{x})) &= f(\mathbf{x}), && \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= g(\mathbf{x}), && \mathbf{x} \in \Gamma_D, \\ (a\nabla u(\mathbf{x})) \cdot \boldsymbol{n} &= g(\mathbf{x}), && \mathbf{x} \in \Gamma_N \end{aligned} \tag{1}$$

and parabolic

$$\begin{aligned} \frac{\partial u(\mathbf{x},t)}{\partial t} - \nabla \cdot (a(\mathbf{x},t)\nabla u(\mathbf{x},t)) &= f(\mathbf{x},t), && (\mathbf{x},t) \in \Omega \times [0,T], \\ u(\mathbf{x},0) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, \\ u(\mathbf{x},t) &= u_D(\mathbf{x},t), && (\mathbf{x},t) \in \Gamma_D \times [0,T], \\ (a\nabla u(\mathbf{x},t)) \cdot \boldsymbol{n} &= g_N(\mathbf{x},t), && (\mathbf{x},t) \in \Gamma_N \times [0,T] \end{aligned} \tag{2}$$

problems are considered. Let us introduce the triangulation $\mathcal{T}_h$ of Ω and the uniform mesh ω_τ with a time step τ in $[0,T]$. The finite element method (FEM) associated with $\mathcal{T}_h$ and the FEM with Crank-Nicholson time stepping on $\mathcal{T}_h \times \omega_\tau$ are used to discretize the problems (1) and (2) respectively. Then, the elliptic problem is reduced to the system

$$K\mathbf{u}_h = \mathbf{f}_h, \tag{3}$$

where K stands for the stiffness matrix. At each Crank-Nicholson time step, the following linear system is to be solved

$$\left(M + \frac{\tau}{2}K\right)\mathbf{u}_h^{n+1} = \left(M - \frac{\tau}{2}K\right)\mathbf{u}_h^n + \tau\mathbf{f}_h^{n+\frac{1}{2}}, \tag{4}$$

where the upper index of the unknown vector indicates the number of the current time step, and M stands for the mass matrix. The modified incomplete Cholesky factorization MIC(0) is used for the preconditioned conjugate gradient (PCG) solution of the systems (3) and (4).

The implementation of two variants of finite elements defined on $\mathcal{T}_h$ is studied, namely, conforming (Courant) and non-conforming (Crouzeix-Raviart) linear finite elements.

We investigate the following three grid generators: Triangle, NETGEN and Gmsh. The quantitative analysis is based on the number of elements/nodes needed to obtain a triangulation of a given domain, satisfying a certain minimal angle condition. Let us remind that the minimal angle directly reflects on the accuracy of the FEM approximation as well as on the condition number of the related stiffness matrix. Some advantages of Triangle are observed in this respect.

The reminder of the paper is organized as follows. Some needed background about MIC(0) factorization is given in the next section. Section 3 contains a condition number analysis of the diagonal compensation. The comparison of the considered mesh generators is summarized in Section 4. Numerical tests for structured (model) and unstructured (general) grids are presented in Section 5. Some concluding remarks are given at the end.

2 MIC(0) Preconditioning

We recall some known facts about the modified incomplete Cholesky factorization MIC(0) [4, 5]. Let $A = (a_{ij})$ be a symmetric $N \times N$ matrix and let

$$A = D - L - L^T, \tag{5}$$

where D is the diagonal and $-L$ is the strictly lower triangular part of A. Then we consider the factorization

$$C_{\mathrm{MIC}(0)} = (X - L)X^{-1}(X - L)^T, \tag{6}$$

where $X = \mathrm{diag}(x_1, \ldots, x_N)$ is a diagonal matrix, such that the sums of the rows of $C_{\mathrm{MIC}(0)}$ and A are equal, i.e.,

$$C_{\mathrm{MIC}(0)}\mathbf{e} = A\mathbf{e}, \quad \mathbf{e} = (1, \ldots, 1) \in \mathrm{I\!R}^N. \tag{7}$$

Theorem 1. *Let $A = (a_{ij})$ be a symmetric $N \times N$ matrix and let*

$$\begin{aligned} L &\geq 0, \\ A\mathbf{e} &\geq 0, \\ A\mathbf{e} + L^T\mathbf{e} &> 0, \quad \textit{where } \mathbf{e} = (1, \ldots, 1)^T. \end{aligned} \tag{8}$$

Then there exists a stable MIC(0) factorization of A, defined by the diagonal matrix $X = \mathrm{diag}(x_1, \ldots, x_N)$, where

$$x_i = a_{ii} - \sum_{k=1}^{i-1} \frac{a_{ik}}{x_k} \sum_{j=k+1}^{N} a_{kj} > 0. \tag{9}$$

It is known, that due to the positive offdiagonal entries of the stiffness matrix K, the MIC(0) factorization is not directly applicable to precondition the FEM system. The diagonal compensation is a simple general approach to avoid this problem.

Remark 1. The numerical tests presented in the last section are performed using the perturbed version of MIC(0) algorithm, where the incomplete factorization is applied to the matrix $\tilde{A} = A + \tilde{D}$. The diagonal perturbation $\tilde{D} = \tilde{D}(\xi) = \mathrm{diag}(\tilde{d}_1, \ldots \tilde{d}_N)$ is defined as follows:

$$\tilde{d}_i = \begin{cases} \xi a_{ii} & \text{if } a_{ii} \geq 2w_i, \\ \xi^{1/2} a_{ii} & \text{if } a_{ii} < 2w_i, \end{cases}$$

where $0 < \xi < 1$ is a parameter and $w_i = \sum_{j>i} -a_{ij}$.

3 Diagonal Compensation: Condition Number Estimate

The stiffness matrix K corresponds to a certain FEM discretization of the elliptic problem (1) on the triangulation $\mathcal{T}_h$. When there are some positive off-diagonal entries in the matrix, the stability conditions for MIC(0) factorization are not satisfied. The diagonal compensation is a general procedure to substitute K by a proper M-matrix $\bar{K}$, to which the MIC(0) factorization is applied. The standard FEM procedure of freezing the coefficient $a(\mathbf{x})$ on each finite element (integral mean value approximation) leads to the consideration of a piece-wise Laplacian problem. The global stiffness matrix reads as

$$K = \sum_{e\in\mathcal{T}_h} K_e, \tag{10}$$

where K_e is the current element stiffness matrix, and the summation sign stands for the FEM assembling procedure. When necessary we will use the notations $K^{(c)}$, $K_e^{(c)}$, $K^{(nc)}$, $K_e^{(nc)}$, where (c) and (nc) indicate the cases of conforming and non-conforming elements. The following important geometric interpretation of the element stiffness matrix $K_e^{(c)}$ is well known (see, e.g., in [3])

$$K_e^{(c)} = t_e \begin{pmatrix} \alpha+\beta & -\alpha & -\beta \\ -\alpha & \alpha+1 & -1 \\ -\beta & -1 & \beta+1 \end{pmatrix}, \tag{11}$$

where $\theta_1 \geq \theta_2 \geq \theta_3 \geq \tau > 0$ are the angles of the triangle $e \in \mathcal{T}$, $a = \cot\theta_1$, $b = \cot\theta_2$, $c = \cot\theta_3$, $\alpha = \dfrac{a}{c}$ and $\beta = \dfrac{b}{c}$. Since $|a| \leq b \leq c$, the element-by-element diagonal compensation is mandatory applied if and only if $a < 0$. Then, the related modified element and global stiffness matrices read respectively as follows

$$\bar{K}_e^{(c)} = t_e \begin{pmatrix} \beta & 0 & -\beta \\ 0 & 1 & -1 \\ -\beta & -1 & \beta+1 \end{pmatrix}, \qquad \bar{K}^{(c)} = \sum_{e\in\mathcal{T}} \bar{K}_e^{(c)}. \tag{12}$$

Note that $\bar{K}_e^{(c)} \equiv K_e^{(c)}$ if $a \geq 0$.

Now, let us turn on to the case of Crouzeix-Raviart non-conforming finite elements. The related test functions are piece-wise linear with interpolation nodes at the mid-points of the sides of the triangle element instead of the vertices (as is for the standard Courant elements). Then, the following important relation is readily seen,

$$K_e^{(nc)} = 4K_e^{(c)}. \tag{13}$$

Theorem 2. *The relative condition number $\kappa(\bar{K}^{-1}K)$ is uniformly bounded by a constant, depending on the minimal angle τ only. More precisely*

$$\kappa = \kappa(\bar{K}^{-1}K) \leq c(\tau) = t^2, \tag{14}$$

where $t = \cot\tau$. The estimate (14) holds for both, conforming and non-conforming linear finite elements.

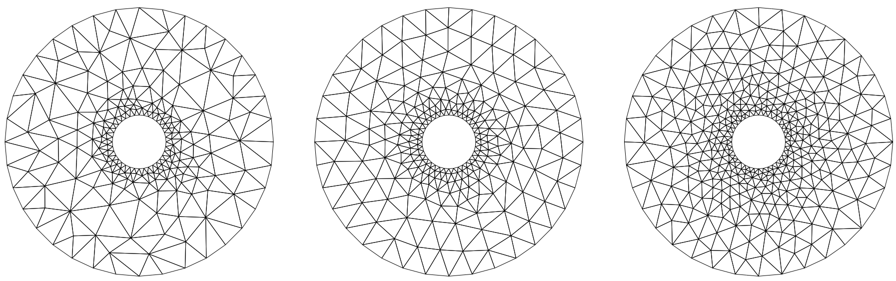
The proof was originally derived for linear conforming finite elements, see [6]. The extension to the case of Crouzeix-Raviart non-conforming elements straightforwardly follows from (13). The important point here is, that the condition number estimate (14) is independent of the size of the discrete problem, as well as, of the number of the positive off-diagonal entries of the global stiffness matrix K. When the Crank-Nicholson scheme is implemented solving the parabolic problem (2) we have to get a preconditioner for the matrix $M + \frac{\tau}{2}K$. Then, a diagonal compensation for K in combination with lumping the mass for M is applied before the MIC(0) factorization. At this point, an advantage of the non-conforming Crouzeix-Raviart elements is, that the related mass matrix is diagonal. This follows easily from the fact, that the quadrature formula on a triangle with nodes in the midpoints of the edges is exact for second degree polynomials.

4 Comparison of Mesh Generators

In this section, we briefly compare the following three mesh generators:

- Triangle (http://www.cs.cmu.edu/˜quake/triangle.html);
- NETGEN (http://www.hpfem.jku.at/netgen/);
- Gmsh (http://geuz.org/gmsh/).

As was shown in the previous section, the minimal angle of the triangulation controls the stability conditions for the MIC(0) factorization. We will also



(a) (b) (c)

Fig. 1. Meshes generated by: (a) Triangle; (b) NETGEN; (c) Gmsh.

Table 1. Mesh properties.

Generator	Minimal angle	Elements	Nodes
Triangle	33.122°	386	229
NETGEN	27.4256°	440	256
Gmsh	31.8092°	688	380

remind that the minimal angle (or the mesh regularity) directly reflects on the accuracy of the FEM approximation as well as on the condition number of the related stiffness matrix. Since a larger minimal angle usually leads to a larger number of elements and nodes in the resulting mesh, it is natural to compare the generators based on the number of elements and nodes, needed to obtain a mesh with a certain minimal angle. The domain we chose for this comparison is a disk. The generated meshes, the related minimal angles, and the numbers of elements and nodes are shown in Figure 1 and Table 1 respectively.

The presented results clearly show that the biggest minimal angle is achieved by Triangle, in which triangulation the number of elements and nodes is the smallest. Triangle is also the only one of the compared generators that accepts the minimal angle as a parameter.

Remark 2. Triangle's documentation states that the algorithm often succeeds for minimum angles up to 33° and usually doesn't terminate for larger angles.

5 Numerical Experiments

The presented numerical tests illustrate the MIC(0)–PCG convergence rate. A relative PCG stopping criteria in the form $\mathbf{r}_k^T C^{-1}\mathbf{r}_k \le \varepsilon^2 \mathbf{r}_0^T C^{-1}\mathbf{r}_0$ is employed. Here $\mathbf{r}_k$ is the residual vector at the k-th iteration and C is the MIC(0) preconditioner. We compare the obtained results in the cases of linear conforming and non-conforming finite elements. The considered model elliptic and parabolic test problems, and the two variants for domains and meshes are given below.

– Elliptic model problem:

$$-\Delta u = f, \qquad \Gamma_D \equiv \partial\Omega,$$

with exact solution

$$u(x,y) = x^3 + y^4 + \sin(y-x).$$

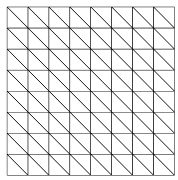
– Parabolic model problem:

$$\frac{\partial u}{\partial t} - \Delta u = f, \qquad \Gamma_D \equiv \partial\Omega,$$

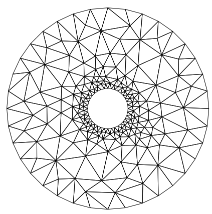
$t \in [0,1], \tau = 0.01$, with exact solution

$$u(x,y,t) = \sin(x-y) + \sin(y-t) + \sin(t-x).$$

– Structured grid:



– Unstructured grid:



The obtained numbers of iterations are presented in both table and graphic form. When parabolic problems are considered, the iteration count is the average value per time step. The asymptotic behavior of the MIC(0)–PCG solver is well expressed in all cases.

Remark 3. A generalized coordinate-wise ordering is used to ensure the conditions for a stable MIC(0) factorization.

5.1 Structured Grids

The results for the model problems on the unit square with uniform structured grids are presented in next two tables. They contain the numbers of iterations for conforming and non-conforming elements respectively.

The convergence rates are graphically presented in Figures 2–3. The solid and dashed lines correspond to the cases of conforming and non-conforming finite elements. The plots give a better opportunity to compare the increase of the iteration counts with the size of the problem. The logarithmic scale more transparently illustrates the asymptotic behavior of the number of iterations. The missing data in the tables (parts of the plots) corresponds to some larger sizes of the discrete problems for which the RAM of the used computer has not been enough.

5.2 Unstructured Grids

The presentation in this section strictly follows the introduced setting from the previous one. Tables 4–5 and Figures 4–5 contain the numerical results for the elliptic and parabolic test problems on the related unstructured grids.

Table 2. MIC(0)–PCG iterations in the unit square: conforming FEM, $\varepsilon = 10^{-6}$.

Mesh	Degrees of Freedom	Elliptic	Parabolic
1	1089	16	5
2	4225	22	7
3	16641	30	10
4	66049	42	13
5	263169	59	16
6	1050625	82	22
7	4198401	115	–

Table 3. MIC(0)–PCG iterations in the unit square: non-conforming FEM, $\varepsilon = 10^{-6}$.

Mesh	Degrees of Freedom	Elliptic	Parabolic
1	800	16	5
2	3136	22	7
3	12416	30	9
4	49408	43	12
5	197120	60	16
6	787456	84	22
7	3147776	119	29

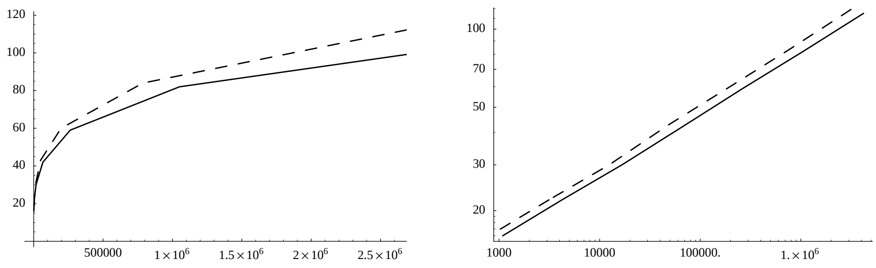


Fig. 2. Number of iterations for the elliptic model problem in the unit square: linear scale (left) and logarithmic scale (right).

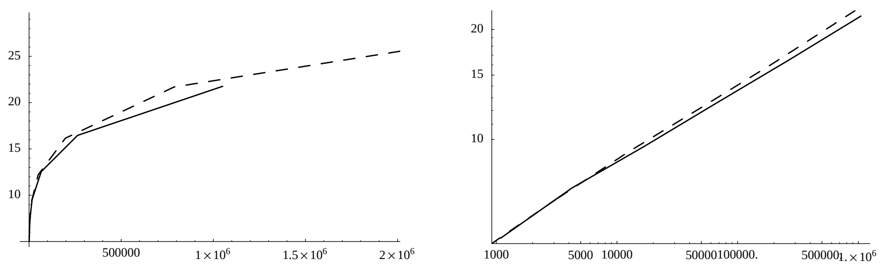


Fig. 3. Number of iterations for the parabolic model problem in the unit square: linear scale (left) and logarithmic scale (right).

5.3 Concluding Remarks

The rigorous theory of MIC(0) preconditioning is applicable only to the model elliptic problem in the unit square when discretized by standard linear conforming finite elements. For this simplest case, the reported number of iterations fully confirms the estimate $n_{it} = O(N^{1/4})$. Here, we observe the same asymptotic of the PCG iterations for all remaining problems, which are not supported by

Table 4. MIC(0)–PCG iterations in the disk: conforming FEM, $\varepsilon = 10^{-6}$.

Mesh	Degrees of Freedom	Elliptic	Parabolic
1	844	17	7
2	3232	26	11
3	12640	40	16
4	49984	58	22
5	198784	79	32
6	792832	106	45
7	3166720	142	–

Table 5. MIC(0)–PCG iterations in the disk: non-conforming FEM, $\varepsilon = 10^{-6}$.

Mesh	Degrees of Freedom	Elliptic	Parabolic
1	615	19	10
2	2388	29	13
3	9408	44	18
4	37344	62	25
5	148800	89	36
6	594048	128	50
7	2373888	179	72

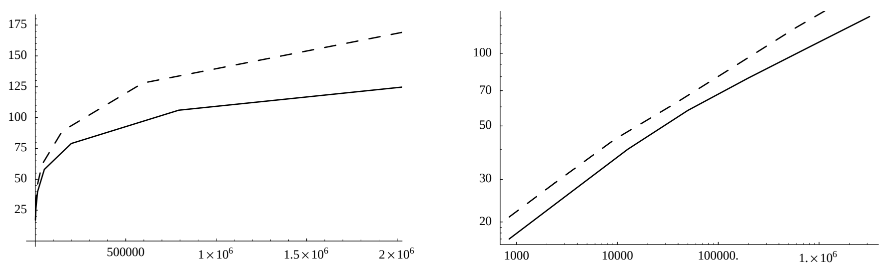


Fig. 4. Number of iterations for the elliptic problem in the disk: linear scale (left) and logarithmic scale (right).

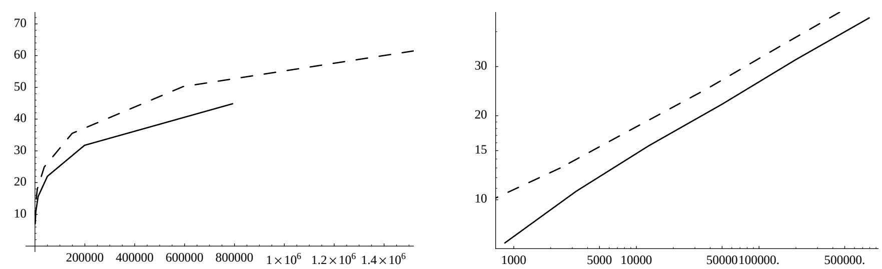


Fig. 5. Number of iterations for the parabolic problem in the disk: linear scale (left) and logarithmic scale (right).

the theory up to now, including the case of Crouzeix-Raviart non-conforming finite elements. As we see, the considered algorithms have a well expressed stable behavior for the unstructured meshes (see Figure 1(a)). The next general conclusion is that the iteration count is smaller for the conforming FEM problems when compared to the results for non-conforming FEM systems of the same size. However, the stable convergence rate of the MIC(0)–PCG solver for Crouzeix-Raviart FEM systems is of a particular importance, due to the special robustness properties of these non-conforming elements.

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