MIC(0) DD Preconditioning of FEM Elasticity Systems on Unstructured Tetrahedral Grids

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Abstract. In this study, the topics of grid generation and FEM applications are studied together following their natural synergy. We consider the following three grid generators: NETGEN, TetGen and Gmsh. The qualitative analysis is based on the range of the dihedral angles of the triangulation of a given domain. After that, the performance of two displacement decomposition (DD) preconditioners that exploit modified incomplete Cholesky factorization MIC(0) is studied in the case of FEM matrices arising from the discretization of the three-dimensional equations of elasticity on unstructured tetrahedral grids.

Keywords: finite element method, preconditioned conjugate gradient method, MIC(0), displacement decomposition.

1 Introduction

Mesh generation techniques are now widely employed in various scientific and engineering fields that make use of physical models based on partial differential equations. While there are a lot of works devoted to finite element methods (FEM) and their applications, it appears that the issues of meshing technologies in this context are less investigated. Thus, in the best cases, this aspect is briefly mentioned as a technical point that is possibly non-trivial.

In this paper we consider the problem of linear elasticity with isotropic materials. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\Gamma = \partial\Omega$ and $\mathbf{u} = (u_1, u_2, u_3)$ the *displacement* in Ω. The components of the *small strain tensor* are

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \le i, j \le 3$$

and the components of the *Cauchy stress tensor* are

$$\tau_{ij} = \sum_{k,l=1}^{3} c_{ijkl}\varepsilon_{kl}(\mathbf{u}), \quad 1 \le i, j \le 3 \ ,$$

where the coefficients c_{ijkl} describe the behavior of the material. In the case of isotropic material the only non-zero coefficients are

$$c_{iiii} = \lambda + 2\mu, \quad c_{iijj} = \lambda, \quad c_{ijij} = c_{ijji} = \mu \ .$$

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Now, we can introduce the *Lamé's* system of linear elasticity (see, e.g., [2])

$$(\lambda+\mu)\sum_{k=1}^{3}\frac{\partial^2 u_k}{\partial x_k \partial x_i}+\mu\sum_{k=1}^{3}\frac{\partial^2 u_i}{\partial x_k^2}+F_i=0, \quad 1\le i\le 3 \tag{1}$$

equipped with boundary conditions

$$\begin{aligned} u_i(\mathbf{x}) &= g_i(\mathbf{x}), \quad \mathbf{x}\in\Gamma_D\subset\partial\Omega \ , \\ \sum_{j=1}^{3}\tau_{ij}(\mathbf{x})n_j(\mathbf{x}) &= h_i(\mathbf{x}), \quad \mathbf{x}\in\Gamma_N\subset\partial\Omega \ , \end{aligned}$$

where $n_j(\mathbf{x})$ denotes the components of the outward unit normal vector $\mathbf{n}$ onto the boundary $\mathbf{x}\in\Gamma_N$. The finite element method (FEM) is applied for discretization of (1) where linear finite elements on a triangulation $\mathcal{T}$ are used. The preconditioned conjugate gradient (PCG) [1] method will be used for the solution of the arising linear algebraic system $K\mathbf{u_h}=\mathbf{f_h}$.

2 MIC(0) DD Preconditioning

We first recall some known facts about the modified incomplete Cholesky factorization MIC(0), see, e.g. [4,5]. Let $A=(a_{ij})$ be a symmetric $n\times n$ matrix and let

$$A=D-L-L^T \ ,$$

where D is the diagonal and $-L$ is the strictly lower triangular part of A. Then we consider the factorization

$$C_{\mathrm{MIC}(0)}=(X-L)X^{-1}(X-L)^T \ ,$$

where $X=\mathrm{diag}(x_1,\ldots,x_n)$ is a diagonal matrix, such that the row sums of $C_{\mathrm{MIC}(0)}$ and A are equal

$$C_{\mathrm{MIC}(0)}\mathbf{e}=A\mathbf{e}, \quad \mathbf{e}=(1,\ldots,1)\in\mathbb{R}^n \ .$$

Theorem 1. *Let $A=(a_{ij})$ be a symmetric $n\times n$ matrix and let*

$$\begin{aligned} L &\ge 0 \\ A\mathbf{e} &\ge 0 \\ A\mathbf{e}+L^T\mathbf{e} &> 0 \quad \textit{where} \quad \mathbf{e}=(1,\ldots,1)^T \ . \end{aligned}$$

Then there exists a stable MIC(0) factorization of A, defined by the diagonal matrix $X=\mathrm{diag}(x_1,\ldots,x_n)$, where

$$x_i=a_{ii}-\sum_{k=1}^{i-1}\frac{a_{ik}}{x_k}\sum_{j=k+1}^{n}a_{kj}>0 \ .$$

It is known, that due to the positive offdiagonal entries of the coupled stiffness matrix K, the MIC(0) factorization is not directly applicable to precondition the FEM elasticity system. Here we consider a composed algorithm based on a separable displacement three-by-three block representation

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \mathbf{u_h} = \mathbf{f_h} \ .$$

In this setting, the stiffness matrix K is spectrally equivalent to the block-diagonal approximations C_{SDC} and C_{ISO}

$$C_{\mathrm{SDC}} = \begin{bmatrix} K_{11} & & \\ & K_{22} & \\ & & K_{33} \end{bmatrix}, \quad C_{\mathrm{ISO}} = \begin{bmatrix} A & & \\ & A & \\ & & A \end{bmatrix} , \tag{2}$$

where $A = \dfrac{1}{3}(K_{11}+K_{22}+K_{33})$. The theoretical background of this displacement decomposition (DD) step is provided by the second Korn's inequality [2]. Now the MIC(0) factorization is applied to the blocks of (2). In what follows, the related preconditioners will be referred to as $C_{\mathrm{SDC\text{-}MIC(0)}}$ and $C_{\mathrm{ISO\text{-}MIC(0)}}$, cf. [2,4,6].

3 Diagonal Compensation

The blocks K_{11}, K_{22}, K_{33} and A correspond to a certain FEM elliptic problem on the triangulation $\mathcal{T}$. Here, we will restrict our analysis to the case of isotropic DD, i.e., we will consider the piece-wise Laplacian matrix

$$A = \sum_{e \in \mathcal{T}} A_e$$

where the summation sign stands for the standard FEM assembling procedure. In the presence of positive offdiagonal entries in the matrix, the conditions of Theorem 1 are not met. To meet these conditions we use *diagonal compensation* to substitute the matrix A by a proper M-matrix $\bar{A}$. After that the MIC(0) factorization is applied to $\bar{A}$. The procedure consists of replacing the positive offdiagonal entries in A with 0 in $\bar{A}$ and adding them to the diagonal, so that $A\mathbf{e} = \bar{A}\mathbf{e}$.

The following important geometric interpretation of the current element stiffness matrix holds (see, e.g., in [7])

$$A_e = \frac{P_e}{6} \begin{bmatrix} \sum\limits_{1 \neq i < j} \ell_{ij} \cot\theta_{ij} & -\ell_{34}\cot\theta_{34} & -\ell_{24}\cot\theta_{24} & -\ell_{23}\cot\theta_{23} \\ -\ell_{34}\cot\theta_{34} & \sum\limits_{2 \neq i < j \neq 2} \ell_{ij}\cot\theta_{ij} & -\ell_{14}\cot\theta_{14} & -\ell_{13}\cot\theta_{13} \\ -\ell_{24}\cot\theta_{24} & -\ell_{14}\cot\theta_{14} & \sum\limits_{3 \neq i < j \neq 3} \ell_{ij}\cot\theta_{ij} & -\ell_{12}\cot\theta_{12} \\ -\ell_{23}\cot\theta_{23} & -\ell_{13}\cot\theta_{13} & -\ell_{12}\cot\theta_{12} & \sum\limits_{i < j \neq 4} \ell_{ij}\cot\theta_{ij} \end{bmatrix} ,$$

where P_e is some constant, depending on the material coefficents, ℓ_{ij} denotes the length of the edge connecting vertices v_i and v_j of the tetrahedron e and θ_{ij} denotes the dihedral angle at that edge. This interpretation shows that each positive offdiagonal entry in the element stiffness matrix corresponds to an obtuse dihedral angle in the tetrahedron e. Also a positive entry tends to infinity when the dihedral angle tends to $180\,^\circ$. In the presence of very large dihedral angles, the relative condition number $\kappa(\bar{A}^{-1}A)$ may become very large. Since the MIC(0) factorization is applied to the auxilary matrix $\bar{A}$, the performance of the preconditioner strongly depends on this relative condifion number. In the two-dimensional case an uniform estimate of the condition number, depending only on the minimal angle was derived (see [6]). In the three-dimensional case, however, it is much harder to obtain an uniform estimate, since the element matrices depend not only on the shape of the elements, but also on elements sizes.

4 Comparison of Mesh Generators

In this section we compare the following three mesh generators:

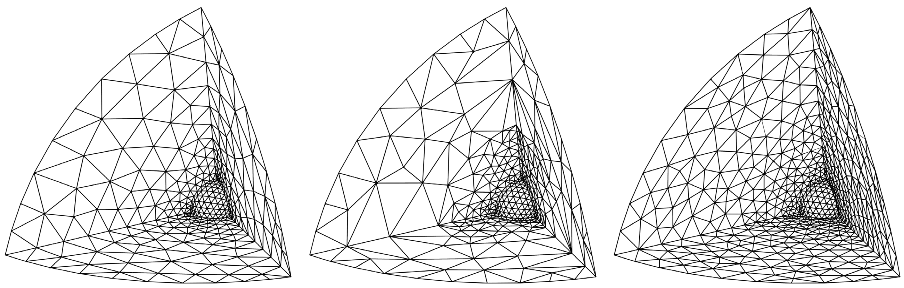
- NETGEN v.4.4 (http://www.hpfem.jku.at/netgen/);
- Tetgen v.1.4.1 (http://tetgen.berlios.de/);
- Gmsh v.2.0.0 (http://geuz.org/gmsh/).

In the previous section we have seen the impact of very large dihedral angles on the preconditioning. Very small and very large angles also affect the accuracy of the FEM approximation as well as the condition number of the related stiffness matrix.

The domain we chose for this comparison is

$$\Omega = \{(x, y, z) \mid 0.1 \le x^2 + y^2 + z^2 \le 1, x, y, z \ge 0\} \ . \tag{3}$$

Different parameters of the grid generators may affect the quality of the resulting meshes. Some generated meshes are shown in Fig. 1 and the minimal and



(a) (b) (c)

Fig. 1. Meshes, generated by: (a) NETGEN; (b) TetGen; (c) Gmsh

Table 1. Resulting Mesh Properties

Generator	Parameters	Min Angle	Max Angle	Elements	Nodes
NETGEN	grading = 1	14.3553 °	151.997 °	436	189
NETGEN	grading = 0.5	19.3608 °	142.821 °	650	245
NETGEN	grading = 0.2	26.1134 °	135.173 °	1882	504
TetGen	ratio = 2	5.06703 °	166.432 °	474	197
TetGen	ratio = 1.5	6.26918 °	169.619 °	714	251
TetGen	ratio = 1.2	6.12442 °	168.717 °	1484	417
Gmsh	$h = 0.05$, $H = 0.5$	13.3345 °	143.297 °	1192	344
Gmsh	$h = 0.03$, $H = 0.3$	20.9614 °	144.173 °	1553	436
Gmsh	$h = 0.015$, $H = 0.15$	18.7442 °	137.373 °	3718	940

maximal angles and numbers of nodes and elements for the three considered mesh generators with various values of the parameters are given in Table 1.

The mesh quality in NETGEN highly depends on the *mesh-size grading* parameter. Decreasing the value of this parameter leads to a mesh with better dihedral angles at the expense of larger number of elements and nodes. In TetGen, the mesh element quality criterion is based on the *minimum radius-edge ratio*, which limits the ratio between the radius of the circumsphere of the tetrahedron and the shortest edge length. It seems, however, that this parameter does not directly reflect on the dihedral angles. With all tested values the resulting meshes contained both very small and very large dihedral angles. For Gmsh, the parameters h and H correspond to the *characteristic lengths*, assigned respectively to the vertices on the inner and the outer spherical boundary of the domain.

The results show that NETGEN generally achieved better dihedral angles than TetGen. Gmsh achieved similar dihedral angles, but with considerably larger number of elements/nodes than NETGEN.

5 Numerical Experiments

The presented numerical test illustrate the PCG convergence rate of the two studied displacement decomposition algorithms. The number of iterations for the CG method are also given for comparison. A relative PCG stopping criterion in the form $\mathbf{r}_k^T C^{-1}\mathbf{r}_k \le \varepsilon^2 \mathbf{r}_0^T C^{-1}\mathbf{r}_0$ is employed. Here $\mathbf{r}_k$ is the residual vector at the k-th iteration and C is the preconditioner.

Remark 1. The experiments are performed using the perturbed version of the MIC(0) algorithm, where the incomplete factorization is applied to the matrix $\tilde{A} = A + \tilde{D}$. The diagonal perturbation $\tilde{D} = \tilde{D}(\xi) = \mathrm{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$ is defined as follows:

$$\tilde{d}_i = \begin{cases} \xi a_{ii} & \text{if } a_{ii} \ge 2w_i \\ \xi^{1/2} a_{ii} & \text{if } a_{ii} < 2w_i \end{cases} ,$$

where $0 < \xi < 1$ is a constant and $w_i = -\sum_{j>i} a_{ij}$.

Table 2. Model Problem in the Unit Cube, $\varepsilon = 10^{-6}$

Mesh	Elements	Nodes	CG	ISO-MIC(0)	SDC-MIC(0)
1	384	125	26	13	13
2	3 072	729	53	17	15
3	24 576	4 913	110	26	22
4	196 608	35 937	192	38	33
5	1 572 864	274 625	459	53	51

Remark 2. A generalized coordinate-wise ordering is used to ensure the conditions for a stable MIC(0) factorization.

Remark 3. Uniform refinement of the meshes in not used in the experiments, since it does not preserve the dihedral angles. For example let us consider the platonic tetrahedron (with dihedral angles $\approx 70.5288\,^\circ$). After splitting it in 8 new tetrahedrons we obtain a mesh with dihedral angles ranging from $54.7356\,^\circ$ to $109.471\,^\circ$. Four of the new tetrahedrons are similar to the original one, and all the other four have one obtuse dihedral angle. The numbers of elements in the experiments with unstructured meshes, thus do not increase exactly 8 times.

5.1 Model Problem in the Unit Cube

We first consider a model pure displacement problem in the unit cube $\Omega = [0,1]^3$ and $\Gamma_D = \partial\Omega$. The material is homogeneous with $\lambda = 1$ and $\mu = 1.5$, and the right-hand side corresponds to the given solution $u_1 = x^3 + \sin(y+z)$, $u_2 = y^3 + z^2 - \sin(x-z)$, $u_3 = x^2 + z^3 + \sin(x-y)$. An uniform initial (coarsest) triangulation with a mesh size $h = 1/4$ is used. The resulting convergence rates are given in Table 2.

5.2 Model Problem in a Curvilinear Domain

We consider the same model problem, but on the domain (3) (see Fig. 1(a)). The resulting convergence rates are given in Table 3. NETGEN is used to generate the meshes for this experiment.

Table 3. Model Problem in the Curvilinear Domain, $\varepsilon = 10^{-6}$

Mesh	Elements	Nodes	CG	ISO-MIC(0)	SDC-MIC(0)
1	1 882	504	54	16	16
2	13 953	3 022	117	17	16
3	107 530	20 589	291	23	21
4	843 040	150 934	715	31	31

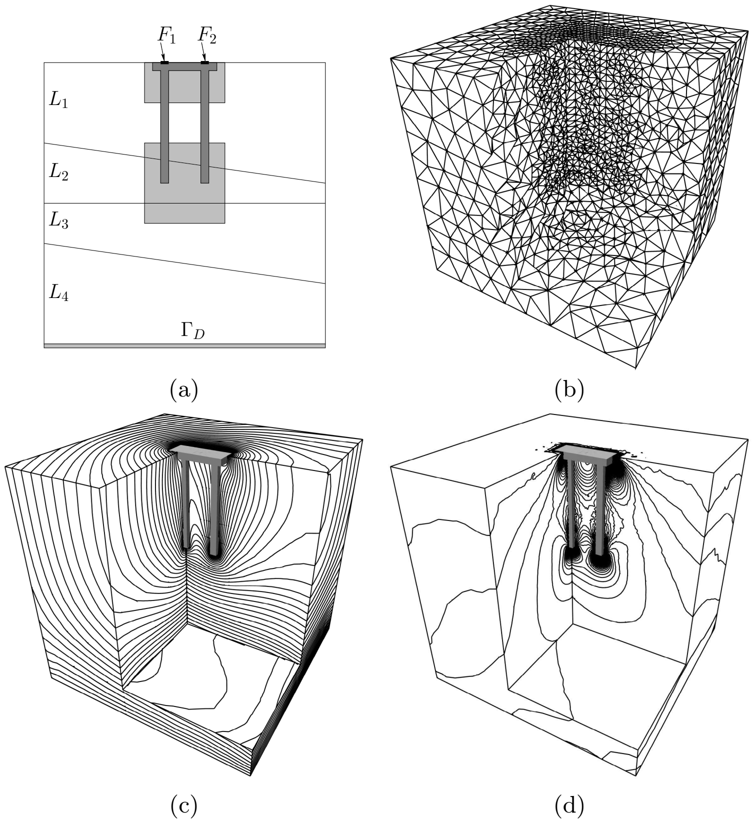


Fig. 2. Pile Foundation System: (a) Geometry; (b) A Mesh with Local Refinement; (c) Vertical Displacements; (d) Vertical Stresses

5.3 Computer Simulation of a Pile Foundation System

We consider the simulation of a foundation system in multi-layer soil media. The system consists of two piles with a linking plate. Fig. 2 (a) shows the geometry of Ω and the related weak soil layers. The generator used here is NETGEN. Meshes are locally refined in areas with expected concentration of stresses, see Fig. 2 (b). The material characteristics of the concrete (piles) are $\lambda_p = 7666.67\,\mathrm{MPa}$, $\mu_p = 11500\,\mathrm{MPa}$. The related parameters for the soil layers are as follows: $\lambda_{L_1} = 28.58\,\mathrm{MPa}$, $\mu_{L_1} = 7.14\,\mathrm{MPa}$, $\lambda_{L_2} = 9.51\,\mathrm{MPa}$, $\mu_{L_2} = 4.07\,\mathrm{MPa}$, $\lambda_{L_3} = 2.8\,\mathrm{MPa}$, $\mu_{L_3} = 2.8\,\mathrm{MPa}$, $\lambda_{L_4} = 1.28\,\mathrm{MPa}$, $\mu_{L_4} = 1.92\,\mathrm{MPa}$. The forces, acting on the top cross-sections of the piles are $F_1 = (150\,\mathrm{kN}, 2000\,\mathrm{kN}, 0)$ and $F_2 = (150\,\mathrm{kN}, 4000\,\mathrm{kN}, 0)$. Dirichlet boundary conditions are applied on the bottom side. Fig. 2 (c) and (d) show contour plots of the solution. Table 4 contains the PCG convergence rate for Jacobi (the diagonal of the original matrix is used as a preconditioner) and the two MIC(0) DD preconditioners.

Table 4. Pile Foundation System, $\varepsilon = 10^{-6}$

Mesh	Elements	Nodes	Jaccobi	ISO-MIC(0)	SDC-MIC(0)
1	24 232	4 389	942	376	307
2	136 955	24 190	1680	564	505
3	859 895	149 111	3150	783	668
4	6 137 972	1 052 306	5416	972	929

5.4 Concluding Remarks

The rigorous theory of MIC(0) preconditioning is applicable to the first test problem only. For a structured grid with a mesh size h and smoothly varying material coefficients, the estimate $\kappa(C_h^{-1}A_h) = O(h^{-1}) = O(N^{1/3})$ holds, where C_h is the SDC-MIC(0) or ISO-MIC(0) preconditioner. The number of PCG iterations in this case is $n_{it} = O(N^{1/6})$. The reported number of iterations fully confirm this estimate. Moreover, we observe the same asymptotics of the PCG iterations for the next two problems, which is not supported by the theory up to now. As we see, the considered algorithms have a stable behaviour for unstructured meshes in a curvilinear domain (see Fig. 1(a)). The robustness in the case of local refinement and strong jumps of the coefficients is well illustrated by the last test problem.

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References

1. O. Axelsson: *Iterative solution methods*, Cambridge University Press, Cambridge, MA, 1994.
2. O. Axelsson, I. Gustafsson: *Iterative methods for the Navier equations of elasticity*, Comp. Meth. Appl. Mech. Engin., **15** (1978), 241–258.
3. O. Axelsson, S. Margenov: An optimal order multilevel preconditioner with respect to problem and discretization parameters, *Advances in Computations, Theory and Practice*, Minev, Wong, Lin eds., **7** (2001), Nova Science: New York, 2–18.
4. R. Blaheta: Displacement Decomposition — incomplete factorization preconditioning techniques for linear elasticity problems, *Numer. Lin. Alg. Appl.*, **1** (1994), 107–126
5. I. Gustafsson: An incomplete factorization preconditioning method based on modification of element matrices, *BIT*, **36** 1 (1996), 86–100.
6. N. Kosturski, S. Margenov, Comparative Analysis of Mesh Generators and MIC(0) Preconditioning of FEM Elasticity Systems, *Numerical Methods and Applications*, T. Boyanov, S. Dimova, K. Georgiev, G. Nikolov eds., *Lecture Notes in Computer Science*, **4310**, Springer, (2007), 74–81.
7. J. Shewchuk: What Is a Good Linear Finite Element? — Interpolation, Conditioning, Anisotropy, and Quality Measures, *Eleventh International Meshing Roundtable* (2002), 115–126.