Displacement Decomposition Circulant Preconditioners for Almost Incompressible 2D Elasticity Systems*

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Abstract. The robustness of the recently introduced circulant block-factorization (CBF) preconditioners is studied in the case of finite element matrices arising from the discretization of the 2D Navier equations of elasticity. Conforming triangle finite elements are used for the numerical solution of the differential problem. The proposed preconditioner M_C is constructed by CBF approximation of the block-diagonal part of the stiffness matrix. In other words, we implement in our algorithm the circulant block-factorization into the framework of the displacement decomposition technique. The estimate $\kappa(M_C^{-1}K) = O\left(\sqrt{\frac{N}{1-\nu}}\right)$ is proved asymptotically on N, where N is the size of the discrete problem. Note, that the corresponding known estimate for the widely used incomplete factorization displacement decomposition preconditioner M_{ILU} is $\kappa(M_{ILU}^{-1}K) = O(\frac{\sqrt{N}}{1-\nu})$.
The theoretical estimate as well as the presented numerical tests show some significant advantages of this new approach for a PCG iterative solution of almost incompressible elastic problems, that is when the modified Poisson ratio ν tends to the incompressible limit case $\nu = 1$.

Key words: almost incompressible elasticity, finite elements, circulant preconditioning, displacement decomposition
AMS subject classifications: 65F10, 65F20, 65N30

1 Introduction

This paper is concerned with the numerical solution of the Navier equations of 2D elasticity problem. Using the finite element method, such a problem is reduced to a linear system of the form $K\mathbf{u} = \mathbf{b}$, where K is a sparse matrix. The considered problem is symmetric and positive definite. We assume also, that K is a large scale matrix. It is well known, that in this case the iterative solvers based on the preconditioned conjugate gradient (PCG) method are the best way

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to solve the linear algebraic system. The key question is how to construct the preconditioning matrix M.

In this paper we consider an application of the recently introduced circulant block–factorization (CBF) algorithm to the plane strain problem of elasticity. The emphasis is on the robustness of the algorithm in the *almost incompressible* case, i.e., when the modified Poisson ratio $\nu \in (0,1)$ tends to the incompressible limit $\nu = 1$.

There are a lot of works dealing with preconditioning iterative solution methods for the FEM elasticity systems. Here we will briefly comment on some of the used approaches. In an earlier paper, Axelsson and Gustafsson [2] have implemented modified point–ILU factorization for this problem. As the coupled system does not lead to an M–matrix, they construct their preconditioners based on the point–ILU factorization of the displacement decoupled block–diagonal part of the original matrix. This approach is known as *displacement decomposition*. It is based on Korn's inequality, and the convergence deteriorates in the almost incompressible case like $O(\frac{1}{\sqrt{1-\nu}})$. The displacement decomposition remains until now one of the most robust approaches (see also, e.g., [3, 7]). Some new block–ILU factorization preconditioners based on block–size reduction are studied in [5]. This factorization exists for symmetric and positive definite block-tridiagonal matrices that are not necessarily M-matrices. Although the approximate factorization is applied to the original matrix, the dependence on ν of the number of iterations remains the same as above, i.e., $O(\frac{1}{\sqrt{1-\nu}})$.

We study in this paper an implementation of the circulant block–factorization (CBF) algorithm as introduced by Lirkov, Margenov and Vassilevski [11], into the framework of the displacement decomposition. The robustness of the algorithm is based on the efficiency of the CBF preconditioners for strongly anisotropic problems (see for more details in [12]). We prove for the new proposed preconditioner M_C the estimate $\kappa(M_C^{-1}K) = O(\sqrt{\frac{N}{1-\nu}})$ asymptotically on N, where N is the size of the discrete problem. Consequently, the growth with ν of the number of iterations is reduced from $O(\frac{1}{\sqrt{1-\nu}})$ to $O(\frac{1}{\sqrt[4]{1-\nu}})$.

The remainder of the paper is organized as follows. Some background facts about the Navier equations of elasticity, their FEM approximation and the related Korn's inequality are presented in §2. The displacement decomposition CBF (DD CBF) algorithm is described in §3. In §4 we give a model problem analysis of the relative condition number of the studied preconditioner. A set of numerical tests illustrating the performance of the resulting preconditioned conjugate gradient algorithm are presented in the last section §5.

2 FEM 2D elasticity equations

We consider in this paper the Dirichlet boundary value plain strain problem of elasticity in the weak formulation of the Navier system of equations. The unknown displacements $\underline{w}^t = (u, v)$ satisfy the following variational equations:

Find $(u, v) \in H_1^0 \times H_1^0$, such that

$$\begin{aligned} a(u,\tilde{u}) + e_{12}(v,\tilde{u}) &= f_1, \\ e_{21}(u,\tilde{v}) + b(v,\tilde{v}) &= f_2, \end{aligned} \qquad \forall(\tilde{u},\tilde{v}) \in H_1^0 \times H_1^0, \tag{1}$$

where $H_1^0 = \{w \in H_1(\Omega) : w|_{\partial\Omega} = 0\}$, and the related bilinear forms are defined by the formulas:

$$a(\phi,\psi) = \int_\Omega \left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{(1-\nu)}{2}\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} \right) d\Omega,$$

$$b(\phi,\psi) = \int_\Omega \left(\frac{(1-\nu)}{2}\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} \right) d\Omega,$$

$$e_{12}(\psi,\phi) = e_{21}(\phi,\psi) = \frac{1+\nu}{2}\int_\Omega \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} d\Omega.$$

Here $\nu \in (0,1)$ stands for the modified Poisson ratio. The notion *almost incompressible* is used for the case $\nu = 1 - \delta$, where δ is a small positive number. Note that if $\nu = 1$ (the material is incompressible), the problem (1) is ill-posed.

Now, let ω be a square mesh, and let Ω be a polygonal domain, triangulated by right isosceles triangles $T \in \tau$ obtained by a diagonal bisection of the square cells of ω.

Let $W = W_1^0 \times W_1^0$, where $W_1^0 \subset H_1^0$ is the finite element space of conforming piecewise linear functions with nodal Lagrangian basis $\{\phi_i\}_{i=1}^N$ corresponding to the triangulation τ. Then the finite element approximation (u^h, v^h) of the problem (1) is determined as follows:

Find $u^h = \sum_{i=1}^N u_i\phi_i$, $v^h = \sum_{i=1}^N v_i\phi_i$, such that

$$\begin{aligned} a(u^h,\phi_i) + e_{12}(v^h,\phi_i) &= f_{1,i}, \\ e_{21}(u^h,\phi_i) + b(v^h,\phi_i) &= f_{2,i}. \end{aligned} \qquad \forall i = 1,\dots,N. \tag{2}$$

Equations (2) are equivalent to the linear system

$$K\underline{w}_h = \underline{b},$$

where K is the stiffness matrix, and $\underline{w}_h = \begin{pmatrix} \underline{u}_h \\ \underline{v}_h \end{pmatrix}$ is the vector of the nodal unknowns $\underline{u}_h = \{u_i\}_{i=1}^N$ and $\underline{v}_h = \{v_i\}_{i=1}^N$.

The stiffness matrix K can be written in the following natural block–structure

$$K = \begin{pmatrix} A & E \\ E^T & B \end{pmatrix},$$

where the blocks A and B correspond to the bilinear forms $a(.,.)$ and $b(.,.)$ respectively.

The following theorem plays a key role in the convergence theory of the displacement decomposition methods.

Theorem 1. *The following Korn's inequality holds*

$$\kappa(K_D^{-1}K) \leq \frac{3+\nu}{1-\nu}, \tag{3}$$

where

$$K_D = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{4}$$

and $\kappa(.)$ stands for the condition number of the matrix.

Proof. It is easy to see, that $\kappa(K_D^{-1}K) \leq \frac{\lambda_4}{\lambda_1}$, where λ_1 and λ_4 are the minimal and the maximal eigenvalue of the generalized eigenvalue problem

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1+\nu}{2} \\ 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 \\ \frac{1+\nu}{2} & 0 & 0 & 1 \end{pmatrix} \underline{w} = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underline{w} \qquad (5).$$

The eigenvalue problem leads to the characteristic equation

$$(1-\lambda)^2 \left(\frac{1-\nu}{2}\right)^2 \left[(1-\lambda)^2 - \left(\frac{1+\nu}{2}\right)^2\right] = 0. \tag{6}$$

The roots of (6) are as follows $\lambda_1 = \frac{1-\nu}{2}$, $\lambda_2 = \lambda_3 = 1$, and $\lambda_4 = \frac{3+\nu}{2}$, that completes the proof of the theorem.

Remark 1 *The Korn's inequality (3) is proved by a different technique in [2].*

3 DD CBF algorithm

We consider in what follows the problem (1) in the unit square, where $\Omega = (0,1) \times (0,1)$ is covered by a uniform square mesh ω, with a size $h = 1/(n+1)$ for a given integer $n \geq 1$.

To define the displacement decomposition circulant block–factorization (DD CBF) preconditioner M_C of the matrix K we consider the auxiliary problem $-au_{xx} - bu_{yy} = f$ with homogeneous Dirichlet boundary conditions, where a and b are positive constants. This problem is discretized by the same finite elements as the original problem (1). This discretization leads to the stiffness matrix G. We assume that the grid points are ordered along the y-lines if $b < a$, and respectively along the x-lines if $a < b$. Then the matrix G can be written in the following form

$$G = tridiag(-G_{i,i-1}, G_{i,i}, -G_{i,i+1}) \qquad i = 1, 2, \ldots, n,$$

where

$$G_{i,i} = tridiag(-g_{j,j-1}, g_{j,j}, -g_{j,j+1}),\ j = (i-1)n+1, \ldots, in,\ i = 1, 2, \ldots, n,$$
$$G_{i,i+1} = diag(g_{j,j+n}),\ j = (i-1)n+1, \ldots, in,\ i = 1, \ldots, n-1,$$
$$G_{i,i-1} = diag(g_{j,j-n}),\ j = (i-1)n+1, \ldots, in,\ i = 2, \ldots, n.$$

The coefficients $g_{i,j}$ are positive and $g_{j,j} \geq g_{j,j-1} + g_{j,j+1} + g_{j,j+n} + g_{j,j-n}$, i.e., the matrix G satisfies the maximum principle.

We will use here the CBF preconditioning for the matrix G as introduced by Lirkov, Margenov and Vassilevski in [11] (see also in [4, 6], [12, 13]). The CBF preconditioner G_{CBF} of the matrix G is defined by

$$G_{CBF} = tridiag(-C_{i,i-1}, C_{i,i}, -C_{i,i+1}) \qquad i = 1, 2, \ldots, n, \tag{7}$$

where $C_{i,j} = Circulant(G_{i,j})$ is a circulant approximation of the corresponding block $G_{i,j}$, defined by a diagonal–by–diagonal averaging of the coefficients. Realizing the CBF algorithm we use exact block LU factorization for the preconditioner G_{CBF}. One important property of the CBF preconditioning is, that the solution of systems with the matrix G_{CBF} requires $O(NlnN)$ arithmetic operations, if FFT is used for factorization of the circulant blocks (see for more details in [9, 11]).

Finally, the DD CBF preconditioner for the stiffness matrix K of the problem (1) is defined by

$$M_C = \begin{pmatrix} A_{CBF} & 0 \\ 0 & B_{CBF} \end{pmatrix}. \tag{8}$$

Obviously, the matrix A corresponds to the auxiliary elliptic problem with $a = 1$ and $b = \frac{1-\nu}{2}$, and respectively the matrix B corresponds to the same differential problem with $a = \frac{1-\nu}{2}$ and $b = 1$. This means that the grid points ordering related to the first diagonal block A is along the y-lines, and contrary the ordering related to the diagonal block B is along the x-lines.

Note, that the above ordering of the unknowns is of key importance for the convergence of the DD CBF preconditioner.

4 Model problem condition number analysis

We will estimate in this section the condition number $\kappa(M_C^{-1}K)$ of the preconditioned system by the DD CBF algorithm.

Theorem 2. *The following inequality holds for the relative condition number of the CBF preconditioner (7)*

$$\kappa(G_{CBF}^{-1}G) < \sqrt{2\epsilon}(n+1) + 2, \tag{9}$$

where $\epsilon = \min\{\frac{b}{a}, \frac{a}{b}\}$.

This estimate is based on the exact solution of the corresponding generalized eigenproblem. A detailed proof of the theorem is presented in [12].

The final result of the model problem condition number analysis is given by the next theorem.

Theorem 3. *The preconditioner M_C defined by the DD CBF algorithm satisfies the estimate*

$$\kappa(M_C^{-1}K) < (3+\nu)\left(\frac{n+1}{\sqrt{1-\nu}} + \frac{2}{1-\nu}\right) \tag{10}$$

Proof. The proof follows directly, applying consequently the Korn's inequality from Theorem 1 and the estimate (9) with $\epsilon = \frac{1-\nu}{2}$, i.e.,

$$\begin{aligned}\kappa(M_C^{-1}K) &\leq \frac{3+\nu}{1-\nu}\max\left(\kappa(A_{CBF}^{-1}A), \kappa(B_{CBF}^{-1}B)\right)\\ &\leq \frac{3+\nu}{1-\nu}\left((n+1)\sqrt{1-\nu}+2\right)\\ &= (3+\nu)\left(\frac{n+1}{\sqrt{1-\nu}}+\frac{2}{1-\nu}\right)\end{aligned}$$

Remark 2 *As a conclusion of the last theorem we get an estimate for the number $n(\varepsilon)$ of the iterations in the PCG algorithm, needed to reduce the relative error with a factor ε, in the form*

$$n(\varepsilon) \leq \sqrt{\left(\frac{n+1}{\sqrt{1-\nu}}+\frac{2}{1-\nu}\right)}\ln\frac{2}{\varepsilon}+1.$$

When $N = n^2$ is large enough, the above estimate can be written in the form $n(\varepsilon) = O\left(\sqrt[4]{\frac{N}{1-\nu}}\right)$.

Remark 3 *Although the presented analysis relates to the model problem in a rectangle, the application of circulants is not limited to this case. An efficient circulant based iterative procedure in L-shaped domain is proposed in [10], where the domain Ω is first transformed to the unit square. Another way to treat problems in domains with more complicated geometry is based on the circulant approximation of the Schur complements in the context of the domain decomposition method.*

5 Numerical tests

We analyze in this section the performance rate of our preconditioned iterative method, varying the size parameter n and the modified Poisson ratio ν. The computations are done with double precision on a SUN Sparc Station.

We recall, that the almost incompressible case corresponds to $\nu = 1-\delta$, where δ is a small positive number.

The Table shows the number of iterations as a measure of the convergence rate of the preconditioners. The iteration stopping criterion is $\|r^{N_{it}}\|/\|r^0\| < 10^{-6}$, where r^j stands for the residual at the jth iteration step of the preconditioned conjugate gradient method.

Asymptotically, the presented data are in a good agreement with the theoretical estimate. The number of iterations has a complex behavior, corresponding to the derived estimate from the last section. One can see how the range, where $n(\varepsilon)$ has a behavior like $n(\varepsilon) = O\left(\sqrt[4]{\frac{N}{1-\nu}}\right)$, grows with n.

Table 1. Number of iterations for the DD CBF preconditioner.

ν	n=32	n=64	n=128	n=256	n=512
0.3	17	21	27	37	49
0.4	18	22	28	37	49
0.5	19	22	29	38	52
0.6	20	25	32	39	54
0.7	22	26	34	41	56
0.8	25	29	37	46	61
0.9	33	37	44	57	71
0.92	36	40	46	60	77
0.94	41	46	52	64	81
0.96	49	53	60	72	91
0.98	67	72	80	91	112
0.99	91	98	106	119	140
0.999	202	266	297	317	344

Remark 4 *Numerical tests for the same problem with pointwise MILU preconditioners are presented in the earlier paper by Axelsson and Gustafsson [2]. Unfortunately these test data are only for coarse grid-sizes with $n \leq 20$ that makes the direct comparision not representative. More recent numerical results are presented in [5], where the block-size reduction ILU preconditioner is applied. Both these preconditioners are characterized by $O(\frac{1}{\sqrt{1-\nu}})$ growth of the number of iterations in the almost incompressible case.*

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