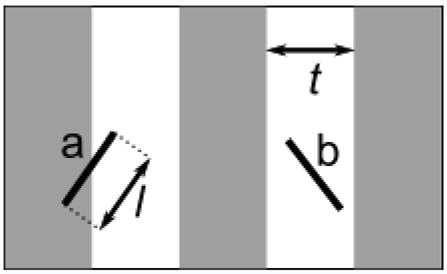
Buffon's needle

In mathematics, **Buffon's needle problem** is a question first posed in the 18th century by Georges-Louis Leclerc, Comte de Buffon: suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

Using integral geometry, the problem can be solved to get a Monte Carlo method to approximate π.

Solution



The "*a*" needle lies across a line, while the "*b*" needle does not.

The problem in more mathematical terms is: Given a needle of length ℓ dropped on a plane ruled with parallel lines *t* units apart, what is the probability that the needle will cross a line?

Let *x* be the distance from the center of the needle to the closest line, let *θ* be the acute angle between the needle and the lines, and let $t \geq \ell$.

The probability density function of *x* between 0 and *t*/2 is

$$\frac{2}{t}\,dx.$$

The probability density function of θ between 0 and π/2 is

$$\frac{2}{\pi}\,d\theta.$$

The two random variables, *x* and *θ*, are independent, so the joint probability density function is the product

$$\frac{4}{t\pi}\,dx\,d\theta.$$

The needle crosses a line if

$$x \le \frac{\ell}{2}\sin\theta.$$

Integrating the joint probability density function gives the probability that the needle will cross a line:

$$\int_{\theta=0}^{\frac{\pi}{2}}\int_{x=0}^{(\ell/2)\sin\theta}\frac{4}{t\pi}\,dx\,d\theta = \frac{2\ell}{t\pi}.$$

For *n* needles dropped with *h* of the needles crossing lines, the probability is

$$\frac{h}{n} = \frac{2\ell}{t\pi},$$

which can be solved for π to get

$$\pi = \frac{2\ell n}{th}.$$

Now suppose $t < \ell$. In this case the probability that the needle will cross a line is

$$\frac{h}{n} = \frac{2\ell}{t\pi} - \frac{2}{t\pi}\left\{\sqrt{\ell^2 - t^2} - t\sec^{-1}\left(\frac{\ell}{t}\right)\right\},$$

Lazzarini's estimate

Mario Lazzarini, an Italian mathematician, performed the Buffon's needle experiment in 1901. Tossing a needle 3408 times, he attained the well-known estimate 355/113 for π, which is a very accurate value, differing from π by no more than 3×10^{-7}. This is an impressive result, but is something of a cheat.

Lazzarini chose needles whose length was 5/6 of the width of the strips of wood. In this case, the probability that the needles will cross the lines is 5/3π. Thus if one were to drop *n* needles and get *x* crossings, one would estimate π as

$$\pi \approx 5/3 \cdot n/x$$

π is very nearly 355/113; in fact, there is no better rational approximation with fewer than 5 digits in the numerator and denominator. So if one had *n* and *x* such that:

$$355/113 = 5/3 \cdot n/x$$

or equivalently,

$$x = 113n/213$$

one would derive an unexpectedly accurate approximation to π, simply because the fraction 355/113 happens to be so close to the correct value. But this is easily arranged. To do this, one should pick n as a multiple of 213, because then $113n/213$ is an integer; one then drops n needles, and hopes for exactly $x = 113n/213$ successes.

If one drops 213 needles and happens to get 113 successes, then one can triumphantly report an estimate of π accurate to six decimal places. If not, one can just do 213 more trials and hope for a total of 226 successes; if not, just repeat as necessary. Lazzarini performed $3408 = 213 \cdot 16$ trials, making it seem likely that this is the strategy he used to obtain his "estimate".