Swendsen-Wang beats Heat-bath

Mario Ullrich

Let $G = (V, E)$ be a graph with finite vertex set V and edge set $E \subseteq \binom{V}{2}$, where $\binom{V}{2}$ is the set of all subsets of V with 2 elements, $N := |V|$. The *Ising model* is defined as the set of possible configurations $\Omega_{\text{IS}} = \{-1, 1\}^V$, together with the probability measure

$$\pi_\beta(\sigma) \;:=\; \pi_\beta^G(\sigma) \;=\; \frac{1}{Z_\beta} \exp\left\{\beta \sum_{\{u,v\}\in E} \mathbf{1}_{\{\sigma(u)=\sigma(v)\}}\right\},$$

where $u \leftrightarrow v$ means that u and v are neighbors in G, Z is the normalization constant and $\beta \geq 0$ is called the inverse temperature.

The goal is to generate an (approximate) sample with respect to π_β in time that is polynomial in the number of vertices N.
This is possible with "local" Markov chains (like the Metropolis algorithm) if β is below some critical value, i.e. $\beta \leq \beta_c$, but impossible for larger values.

We prove that the mixing time of the Swendsen-Wang algorithm, which is possibly the most widely used algorithm to sample from such distributions, cannot be much larger than the mixing time of any local Markov chain if the maximum degree of the underlying graph is bounded. I.e. if the maximal degree of G is Δ, then

$$\lambda(P_{\text{SW}}) \;\geq\; c^{\Delta}\, \lambda(P_{\text{L}}),$$

where P_{SW} (resp. P_{L}) is the transition matrix of the Swendsen-Wang (resp. any local) Markov chain, $\lambda(\cdot)$ denotes the spectral gap and c is some constant that depends only on the temperature.

As a consequence we get rapid mixing of the Swendsen-Wang algorithm for the two-dimensional Ising model at high temperatures and for graphs with maximal degree Δ if $\beta < \Delta^{-1}$. This improves upon the known results on the mixing time of Swendsen-Wang.

Furthermore we present a modified version of the Swendsen-Wang chain that is applicable for planar graphs G and prove that it is rapidly mixing at all temperatures.
This seems to be the first Markov chain for the Ising model that is proved to mix in polynomial time for all temperatures.