Approximating Stochastic Integrals in L_p

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Consider a square integrable random variable Z of the form

$$Z = f(W_1) = g(S_1),$$

where W is a Brownian motion and S a geometric Brownian motion. Discretize the stochastic integral in

$$Z - \mathbb{E}Z = \int_0^1 \phi(t, W_t) dW_t = \int_0^1 \psi(t, S_t) dS_t$$

along a time net $\tau^n = (\tau_i)_{i=0}^n$ consisting of stopping times, and look at the approximation error

$$C_X(Z, \tau^n) = Z - \mathbb{E}Z - \sum_{i=1}^{n} v_{i-1}(X_{\tau_i} - X_{\tau_{i-1}}),$$

where $X = W$ or $X = S$ and v_{i-1} is a random variable measurable w.r.t. $\mathcal{F}_{\tau_{i-1}}$ with $\mathcal{F}_t = \overline{\sigma(W_s : s \leq t)}$.

We discuss equivalent conditions regarding the L_p convergence rate of the error as $n \to \infty$, properties of the time nets τ^n, and fractional smoothness of f in terms of Malliavin Besov spaces or a Riemann-Liouville type operator.

In particular, for a large class of functions f, we find sufficient conditions on deterministic time nets to obtain the optimal convergence rate

$$\|C_X(Z, \tau^n)\|_p \leq \frac{c}{\sqrt{n}}.$$

Convergence with respect to a stronger norm than the usually employed L_2 norm provides better tail estimates, and mathematically, the step is interesting because the orthogonal structure of L_2 is not available in L_p, $p > 2$.

These kinds of convergence results can be applied, for example, to simulation, or to stochastic finance: in option pricing, the approximation error corresponds to the hedging error appearing when a theoretical, continuously rebalanced portfolio is replaced by another one with only finitely many trading times.