Collision Number Statistics For Transport Processes

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Physical observables are often represented as walkers performing random displacements. When these walkers undergo a limited number of collisions before leaving the explored domain, the diffusion approximation might lead to incongruous results. In this presentation we derive an explicit formula for the moments of the number of particle collisions in an arbitrary volume, for a broad class of transport processes. This approach is shown to generalize the celebrated Kac formula for the moments of residence times, which is recovered in the diffusion limit, i.e., for a large collision number. Some applications are illustrated for bounded, unbounded and absorbing domains.

Many practical problems, encompassing areas as diverse as research strategies, market evolution, percolation through porous media, and DNA translocation through nanopores, to name only a few, demand assessing the statistics of the random residence time $t_{\mathcal{V}}$ spent by a walker inside a given domain $\mathcal{V}$. Indeed, complex physical systems are often described in terms of 'particles' undergoing random displacements, resulting either from the intrinsic stochastic nature of the underlying process, or from uncertainty. As a particular case, when the particle is lost upon touching the boundary $\partial\mathcal{V}$ of $\mathcal{V}$, the residence time is usually called first-passage time. Fully characterizing $t_{\mathcal{V}}$ is an awkward task, since its distribution generally depends on walker dynamics, geometry, boundaries and initial conditions, so that one has often to be content with the mean residence time. This has motivated a large number of theoretical investigations over the last decade, covering both homogeneous and heterogeneous, scale-invariant media. In the former case, the dynamics of the walker is usually modelled by regular Brownian motion, whereas in the latter one resorts to anomalous diffusion. A seminal work developed by Kac, based on a path integral approach, allows all the moments of the residence times of Brownian particles to be evaluated by resorting to convolutions over the ensemble equilibrium distribution of the walkers, for arbitrary boundary conditions on $\mathcal{V}$. However, in many realistic situations, the walker typically undergoes a limited number of collisions before leaving the explored domain, so that the diffusion limit is possibly not attained. Examples are widespread, and arise in, e.g., gas dynamics, neutronics and radiative transfer, electronics, and biology. In all such systems, the stochastic path can be thought of as a series of straight-line flights, separated by random collisions, and the dynamics is better described in terms of the Boltzmann equation, rather than the (anomalous) Fokker-Planck equation. A natural variable for describing the walker evolution is therefore the number of collisions $n_{\mathcal{V}}$ within the observed volume. Application of the diffusion approximation to the characterization of the counting statistics, which amounts to assuming a large number of collisions in $\mathcal{V}$, might lead to inaccurate results. In the present work, we address the issue of generalizing Kac approach to random walkers obeying the Boltzmann equation, i.e., not satisfying the diffusion regime, for arbitrary geometries and boundary conditions. We derive an explicit formula for the moments of $n_{\mathcal{V}}$, and illustrate its relation to the equilibrium distribution of the walkers. Knowledge of higher order moments allows estimating the uncertainty on the average, as well as reconstructing the full distribution of the collision number. We show that when $\mathcal{V}$ is large as compared to the typical size of a flight, so that the diffusion limit is reached, Kac formula is recovered.