Monte Carlo Methods Based on Analytic Extension of the Resolvent of the Helmholtz Equation[1]

Vitaliy Lukinov

Institute of Computational Mathematics and Mathematical Geophysics SB RAS

Eighth IMACS Seminar on Monte Carlo Methods
August 29 – September 2, 2011, Borovets, Bulgaria

[1] This work was supported by RFBR, grant 09-01-00639, RFBR, grant 11-01-00252 and NSU

Agenda

Helmholtz equation

Let us consider the Dirichlet problem for the Helmholtz equation in a domain $D \subset R^3$ with boundary Γ:

$$(\Delta + c)u = -g, \quad u|_\Gamma = \varphi \tag{1}$$

Let us assume that the following conditions hold. The function g is satisfied Holder condition in $\overline{D}$, D is a bounded open set in R^3 with a regular boundary Γ, the function φ is continuous on Γ, $c < c^*$, where $-c^*$ is the first eigen value of Laplace operator defined on the domain D.

Outlook

The solution as Neumann series

It is well known that solution to the problem (1) satisfies the integral equation

$$u = \int_D k(r, r'; c) u(r') dr' + h(r), \text{or } u = K_c u + h, \qquad (2)$$

One of approaches to constructing statistical algorithms for solving (2) is based on the following presentation

$$u = (I - K_c)^{-1} h = h + R_c h, \; R_c = \sum_{i=1}^{\infty} K_c^i \qquad (3)$$

Eigenvalue algorithm

Let us consider the following problem

$$Ku = \lambda u$$

We can use well known formula to calculate first eigenvalue

$$\lambda_1 = \lim_{i \to \infty} \frac{a_i}{a_{i+1}}$$

$$R_c h = \sum_{i=0}^{\infty} a_i c^i, \; a_i = K^{i+1} h$$

Eigenvalue algorithm

Using the same methods by parametric derivatives. We consider the parametric derivatives $u^{(m)}$ of the solution to the particular problem [Mikhailov G.A.]

$$(\Delta + c)u = 0, \quad u|_{\Gamma} = 1$$

It obviously, that

$$(\Delta + c)u^{(1)} = -u, \quad u^{(1)}|_{\Gamma} = 0$$

$$(\Delta + c)u^{(m)} = -mu^{(m-1)}, \quad u^{(m)}|_{\Gamma} = 0, \; m = 1, 2, ...$$

Therefore

$$c^* - c = \lim_{m \to \infty} \frac{mu^{(m-1)}}{u^{(m)}}$$

Outlook

Probabilistic representation

It is known (see [Elepov B.S., Mikhailov G.A.(1969)]),that under above stated conditions the probability representation for the solution to problem (1) has the form

$$u(x) = \mathrm{E}[\int_0^{\tau} e^{s(t;c)} g(\xi_t) dt + e^{s(\tau;c)} \varphi(\xi_\tau)], \quad s(t;c) = \int_0^{t} c(\xi_{t'}) dt', \tag{4}$$

where ξ_t is the diffusion process corresponding to the Laplace operator and starting at the point x, τ is the moment of the first exit of the process from the domain D.

Outlook

The convergence of Neumann series

Let us consider the remainder of Neumann series

$$Rem_n = \sum_{i=n+1}^{\infty} a_i c^i$$

When u(r,c) is meromorphic function we have the following estimate [Kublanovskaya V.N.,1959]

$$|Rem_n| = O(\delta^{n+1} n^{r-1})$$

Here $\delta = \left| c/c_1^{(k)} \right|$, $c_1^{(k)}$ is any of the poles lying on the convergence circle, r is the highest multiplicity of this poles

Outlook

The relative error of the eigenvalue algorithm

Now note that

$$\varepsilon_i = \left|(\frac{a_i}{a_{i+1}} - \lambda_1)/\lambda_1\right|$$

In this case

$$\varepsilon_i = O(\delta_1^{n+1}),$$

where $\delta_1 = |\lambda_1/\lambda_2|$, when λ_1, λ_2 are the simple poles.

$$\varepsilon_i = O(\delta_1^{n+1} i^{r-1}),$$

when λ_1 is the simple pole, λ_2 is the multiple pole (r is the multiplicity of pole).

Map of spectral parameter

$$\varphi : \eta \to c, \quad \varphi(\eta) = \sum_{i=1}^{\infty} s_i \eta^i.$$

$$F(\eta) = u(\varphi(\eta), r) = \sum_{i=0}^{\infty} b_i \eta^i,$$

where

$$b_i = \sum_{k=0}^{i} a_k d_k^{(i)}, \quad d_k^{(i)} = \frac{1}{i!} \left(\frac{\partial^i}{\partial \eta^i} (\varphi(\eta))^k \right)_{\eta=0}$$

$$u(c,r) = u(\varphi(\varphi^{-1}(c)), r) = F(\eta) = \sum_{i=0}^{\infty} b_i \eta^i =$$

$$= \sum_{i=0}^{\infty} \eta^i \sum_{k=1}^{i} d_k^{(i)} a_i = \sum_{k=1}^{\infty} a_i l_i,$$

where $l_i = \sum_{k=i}^{\infty} d_k^{(i)} \eta^i$

$$u(c,r) = \mathsf{E}[\int_0^\tau e^{\varphi(\eta)t} g(\xi_t)dt + e^{\varphi(\eta)\tau}\phi(\xi_\tau)]$$

$$e^{\varphi(\eta)t} = \sum_{i=0}^{\infty} \frac{\varphi^i(\eta)t^i}{i!} = \sum_{i=0}^{\infty} b_i \eta^i,$$

where $b_i = \sum_{k=1}^{i} d_k^{(i)} \frac{t^k}{k!}$

$$u(c,r) \approx \sum_{k=0}^{\infty} \mathsf{E}[\int_0^\tau \frac{t^k}{k!} \left(\sum_{n=k}^{m} d_k^{(n)} \eta^n \right) g(\xi_t)dt + e^{\varphi(\eta)\tau}\phi(\xi_\tau)]$$

Certain maps

- Kublanovskaya V.N.,Sabelfeld K.K.

$$c = \varphi(\eta) = \frac{4\alpha\eta}{(1-\eta)^2}$$

- Mikhailov G.A.

$$c = \varphi(\eta) = \frac{\alpha}{(1-\beta\eta)}$$

- $$c = \varphi(\eta) = \alpha + \beta \arctan \eta$$

Outlook

Metaharmonic equation

Let us consider the following problem

$$\begin{cases} (\Delta + c)^{p+1} u = -g, \\ (\Delta + c)^k u|_\Gamma = \varphi_k, \quad k = 0, \dots, p. \end{cases} \tag{5}$$

Metaharmonic equation

Theorem (Mikhailov G.A.,Lukinov V.L., 2004)

Under above stated conditions, the p-th parametric derivative of the solution to problem (1) with the functional parameters

$$\varphi = \sum_{k=0}^{p} \frac{(-1)^k (c - c_1)^{p-k}}{p!} \varphi_k, \quad g_1 = \frac{(-1)^p}{p!} g \tag{6}$$

is a solution to problem (1) (all derivatives are calculated at the point c_1). Here p-th parametric derivative satisfies the following problem

$$\begin{cases} (\Delta + c)^{p+1} u = -g \frac{c_0^{i+1}}{i!}, \\ (\Delta + c)^k u|_\Gamma = \varphi \frac{c_0^{i+1}}{i!}, \quad k = i-1, \\ (\Delta + c)^k u|_\Gamma = 0, \quad k = i-2, \dots, 0. \end{cases} \tag{7}$$

Metaharmonic equation

Theorem (2)

If $c < c^*$ and the first spatial derivatives of the functions $\{u_k^{(i)}\}$, $i = 1, \ldots, p+1$, are uniformly bounded in $\bar{D}$, then

$$|u(r) - \mathrm{E}\tilde{\eta}_{1,\varepsilon}^{(p)}| \leq C_p \varepsilon, \quad r \in D, \quad \varepsilon > 0,$$

where

$$\tilde{\tilde{\eta}}_{1,\varepsilon}^{(p)} = \sum_{i=0}^{N} \left\{ \left[\prod_{j=0}^{i-1} s(c, d_j) \right] g(\rho_i) \frac{d_i^2 G(\rho; c, d_i)}{2nG(\rho; 0, d_i)} \right\}^{(p)} \tag{8}$$

$$+ \left\{ \left[\prod_{j=0}^{N-1} s(c, d_j) \right] \varphi(r_N, c) \right\}^{(p)},$$

Metaharmonic equation

Here, $d_j = d(r_j)$, $D(r_i)$ is a ball of radius d_i with the center at the point r_i, $G(\rho; c, d)$ is the sperical Green's fuction; and

$$s(c,d) = \begin{cases} d\sqrt{c}/\sin\left(d\sqrt{c}\right), & c \geq 0; \\ d\sqrt{|c|}/\sinh\left(d\sqrt{|c|}\right), & c \leq 0. \end{cases}$$

The practical estimates

$$\tilde{\tilde{\eta}}^{(p)}_{1,\varepsilon} = \sum_{i=0}^{N} \left\{ \left[\prod_{j=0}^{i-1} \frac{\sqrt{c} d_j}{\sin(\sqrt{c} d_j)}\right] g(\nu_i, \omega_i) \frac{d_i^3 \sqrt{c} \sin(\sqrt{c}(d_i - \nu_i))}{6(d_i - \nu_i)\sin(\sqrt{c} d_i)} \right.$$

$$\left. + \left\{ \left[\prod_{j=0}^{N-1} \frac{\sqrt{c} d_j}{\sin(\sqrt{c} d_j)}\right] \varphi(r_N, c) \right\}^{(p)} \right.$$

The practical estimates

$$\eta_k = \sum_{p=0}^{k}\Bigg[\sum_{i=0}^{N}\Bigg\{\Bigg[\prod_{j=0}^{i-1}\frac{\sqrt{c}d_j}{\sin(\sqrt{c}d_j)}\Bigg]\frac{c_0^{i+1}}{i!}g(\nu_i,\omega_i)\frac{d_i^3\sqrt{c}\sin(\sqrt{c}(d}{6(d_i-\nu_i)\sin(}$$

$$+\Bigg\{\Bigg[\prod_{j=0}^{N-1}\frac{\sqrt{c}d_j}{\sin(\sqrt{c}d_j)}\Bigg]\frac{c_0^{i+1}}{i!}\varphi(r_N,c)\Bigg\}^{(p)}\Bigg]$$

The random variables ν_i distributed in the interval $(0, d_i)$ with the probability density $6x(1-x/d_i)d_i^{-2}$ and the unit isotropic vectors ω_i are modeled by means of the well-known formulas
[S. M. Ermakov and G. A. Mikhailov, 1982].

Outlook

Lemma

Under above stated conditions, and $c < c^*$. Then

$$\frac{\partial^p u}{\partial c^p} = u^{(p)} = \mathsf{E}[\int_0^\tau t^p e^{ct} g(\xi_t) dt + \sum_{l=0}^{p} C_p^l \tau^{p-l} e^{ct} \frac{\partial^l}{\partial c^l} \varphi(\xi_\tau, c)].$$

On the other hand we have

$$u(c,r) = \mathsf{E}[\int_0^\tau e^{ct} g(\xi_t)dt + e^{c\tau}\varphi(\xi_\tau)] =$$

$$= \sum_{i=0}^{\infty} \mathsf{E}[\int_0^\tau \frac{c^i t^i}{i!} g(\xi_t)dt + \frac{c^i \tau^i}{i!}\varphi(\xi_\tau)] = \sum_{i=0}^{\infty} c^i a_i,$$

where $a_i = \mathsf{E}[\int_0^\tau \frac{t^i}{i!} g(\xi_t)dt + \frac{\tau^i}{i!}\varphi(\xi_\tau)]$;

$R_n = \sum_{i=n}^{\infty} c^i a_i = O(|c/c^*|^{n+1} n^{r-1})$.

Outlook

Covariance function

The oscillations of the plate in a bounded region $D \subset R^2$ under action of the random field stress $\sigma(r) = -g(r)$ describes equation of the form

$$\Delta^2 u + cu = -g, \quad u|_\Gamma = \varphi_0, \quad \Delta u|_\Gamma = \varphi_1, \tag{9}$$

thus it is possible consider the randomness of the boundary functions $\varphi_0(r)$ and $\varphi_1(r)$ [Bolotin V.V.,1979] also.
The solution to this problem is a random field. Required to determine its covariance function $v(r, r') = \mathrm{E}[u(r)u(r')]$.
It is assumed that $\mathrm{E}g(r) \equiv \mathrm{E}\varphi_0(r) \equiv \mathrm{E}\varphi_1(r) \equiv 0$ and, hence, $\mathrm{E}u(r) \equiv 0$.

Solution $u(r, c, g, \varphi_0, \varphi_1)$ can be approximately replaced the corresponding partial sum of Loran series. To construct estimates of the unknown parameter derivatives of $u^{(1)}(r, 0, g, \varphi_0, \varphi_1)$, $u^{(2)}(r, 0, g, \varphi_0, \varphi_1)$ we use the following way. Differentiating with respect to the parameter c equation (??), we get for $c = 0$:

$$\Delta^2 u^{(1)} = -u, \quad u^{(1)}\Big|_\Gamma = \Delta u^{(1)}\Big|_\Gamma = 0. \tag{10}$$

Hence it is clear that the parametric derivative $u^{(1)}(r, 0, g, \varphi_0, \varphi_1)$ is the solution of the following problem:

$$\begin{cases} \Delta^4 u = -g, \quad u\Big|_\Gamma = 0, \quad \Delta u\Big|_\Gamma = 0, \\ \Delta^2 u\Big|_\Gamma = -\varphi_0, \quad \Delta^3 u\Big|_\Gamma = -\varphi_1. \end{cases} \tag{11}$$

The parametric derivative $u^{(2)}(r, 0, g, \varphi_0, \varphi_1)/2$ is the solution of the following problem:

$$\begin{cases} \triangle^6 u = g, \quad \triangle^k u\Big|_\Gamma = 0, \quad k = 0, \ldots, 3, \\ \triangle^4 u\Big|_\Gamma = \varphi_0, \quad \triangle^5 u\Big|_\Gamma = \varphi_1 \end{cases} \tag{12}$$

Estimates for solutions of (??), (??) respectively are as follows:

$$\tilde{\tilde{\eta}}_{1,\varepsilon}^{(3)} = \sum_{i=0}^{N} \left\{ \sum_{k=0}^{3} \mathrm{C}_3^{\mathrm{k}} S_i^{(3-k)}(0) \frac{G^{(k)}(\rho;0,d_i)}{G(\rho;0,d_i)} \right\} \frac{[-d_i^2 g(\rho_i)]}{24} + S_N^{(3)}(0)\frac{\varphi_1(r_N)}{6} - S_N^{(2)}(0)\frac{\varphi_0(r_N)}{2}, \tag{13}$$

$$\tilde{\tilde{\eta}}_{1,\varepsilon}^{(5)} = \left\{ \sum_{i=0}^{N} \sum_{k=0}^{5} \mathrm{C}_5^{\mathrm{k}} S_i^{(5-k)}(0) \frac{G^{(k)}(\rho;0,d_i)}{G(\rho;0,d_i)} \right\} \frac{d_i^2 g(\rho_i)}{480} - S_N^{(5)}(0)\frac{\varphi_1(r_N)}{120} + S_N^{(4)}(0)\frac{\varphi_0(r_N)}{24}, \tag{14}$$

where $S_i(c) = \prod_{j=0}^{i-1} s(c, d_j)$.

Representation of estimates shows that it is necessary to find values derivatives of $S_i^{(k)}(0)$ for $k=\overline{0,3}$. Differentiating with respect to parameter c the logarithmic derivative of $S^{(1)}=S(\ln S)^{(1)}$, we can easily obtain the following relations

$$S^{(2)}=S^{(1)}(\ln S)^{(1)}+S(\ln S)^{(2)},$$

$$S^{(3)}=S^{(2)}\ln S^{(1)}+2S^{(1)}(\ln S)^{(2)}+S(\ln S)^{(3)},$$

$$S^{(4)}=S^{(3)}\ln S^{(1)}+3S^{(2)}(\ln S)^{(2)}+3S^{(1)}(\ln S)^{(3)}+S(\ln S)^{(4)},$$

$$S^{(5)}=S^{(4)}\ln S^{(1)}+4S^{(3)}(\ln S)^{(2)}+6S^{(2)}(\ln S)^{(3)}$$
$$+4S^{(1)}(\ln S)^{(4)}+S(\ln S)^{(5)}.$$

Derivatives of $(\ln S)^{(k)},\ k=\overline{1,5}$ at $c=0$ is easily obtained using equations

$$s(c,d) = 1 + \frac{d^2}{4}c + \frac{3d^4}{2^6}c^2 + \frac{19d^6}{2^8 3^2}c^3 + \frac{211}{2^{14}3^2}c^4 + \frac{1217}{2^{16}\cdot 3\cdot}$$

$$(\ln S_i)^{(1)} = \sum_{j=0}^{i-1} \frac{s^{(1)}(0,d_j)}{s(0,d_j)} = \sum_{j=0}^{i-1} \frac{d_j^2}{4},$$

$$(\ln S_i)^{(2)} = \sum_{j=0}^{i-1} \left\{ \frac{s^{(2)}(0,d_j)}{s(0,d_j)} - \left(\frac{s^{(1)}(0,d_j)}{s(0,d_j)} \right)^2 \right\} = \sum_{j=0}^{i-1} \frac{d_j^4}{2^5},$$

$$(\ln S_i)^{(3)} = \sum_{j=0}^{i-1} \left\{ \frac{s^{(3)}(0,d_j)s(0,d_j) - s^{(2)}(0,d_j)s^{(1)}(0,d_j)}{s^2(0,d_j)} - \right.$$

$$\left. - \frac{2s^{(1)}(0,d_j)[s^{(2)}(0,d_j)s(0,d_j) - (s^{(1)}(0,d_j))^2]}{s^3(0,d_j)} \right\} =$$

$$(\ln S_i)^{(4)} = \sum_{j=0}^{i-1} \left\{ \frac{s^{(4)}(0,d_j)}{s(0,d_j)} - \frac{4s^{(3)}(0,d_j)s^{(1)}(0,d_j)}{s^2(0,d_j)} \right.$$

Using the following asymptotic relations

$$\frac{J_0(z\sqrt{c})}{J_0(d\sqrt{c})} \sim 1 + \frac{d^2 - z^2}{4}c + \frac{3d^4 - 4z^2d^2 + z^4}{2^6}c^2$$
$$+ \frac{19d^6 - 3^3z^2d^4 + 3^2z^4d^2 - z^6}{2^8 3^2}c^3$$
$$+ \frac{211d^8 - 19\cdot 2^4d^6z^2 + 3^3 2^2 d^4z^4 - 2^4d^2z^6 + z^8}{2^{14}3^2}c^4$$
$$+ \left[\frac{3\cdot 1217d^{10} - 211\cdot 5^2d^8z^2 + 19\cdot 2^2\cdot 5^2d^6z^4}{2^{16}\cdot 3^2\cdot 5^2}\right.$$
$$\left. + \frac{-2^2\cdot 3\cdot 5^2d^4z^6 + 5^2d^2z^8 - z^{10}}{2^{16}\cdot 3^2\cdot 5^2}\right]c^5$$

$$N_0^{(2)}(z\sqrt{c}) \sim \frac{2}{\pi}\left[-\frac{1}{2c^2} - \frac{z^2}{2^3c} - \frac{3z^4}{2^7} + \left(\gamma + \ln\frac{z\sqrt{c}}{2}\right)\frac{z^4}{2^5}\right],$$

$$N_0^{(3)}(z\sqrt{c}) \sim \frac{2}{\pi}\left[\frac{1}{c^3} + \frac{z^2}{2^3c^2} + \frac{z^4}{2^6c} + \frac{z^6}{2^8 3^2} - \left(\gamma + \ln\frac{z\sqrt{c}}{2}\right)\frac{z^6}{2^7 3 \ldots}\right.$$

$$N_0^{(4)}(z\sqrt{c}) \sim \frac{2}{\pi}\left[-\frac{3}{c^4} - \frac{z^2}{4c^3} - \frac{z^4}{2^6c^2} - \frac{z^6}{2^8 3c} - \frac{25z^8}{2^{14}3^2} + \left(\gamma + \ln \ldots\right.\right.$$

we derive that

$$G^{(2)}(\rho;0,d)=\frac{1}{2^8\pi}(6z^4-16z^2d^2+10d^4+4z^4\ln\frac{d}{z}),$$

$$G^{(3)}(\rho;0,d)=\frac{1}{2^{10}3^2\pi}(175d^6-270d^4z^2+108d^2z^4-13z^6-\ \ldots$$

$$G^{(4)}(\rho;0,d)=\frac{1}{2\pi}(\frac{677d^8}{2^{13}3}+\frac{25z^8}{2^{13}3^2}-\frac{z^6d^2}{2^73}-\frac{23d^6z^2}{2^63^2}+\frac{15z^4d^4}{2^{10}}\ \ldots$$

$$G^{(5)}(\rho;0,d)=\frac{1}{2\pi}(\frac{71\cdot 103d^{10}}{2^{12}\cdot 3\cdot 5^2}-\frac{137z^{10}}{2^{15}\cdot 3^2\cdot 5^2}+\frac{5z^8d^2}{2^{13}3}-\frac{5\cdot 67\ldots}{2^{1}\ldots}\ \ldots$$

$$-\frac{5^2z^6d^4}{2^{12}3}+\frac{5\cdot 23z^4d^6}{2^{10}3^2}-\frac{z^{10}}{2^{13}\cdot 3\cdot 5}\ln\frac{d}{z}).$$

$$
\begin{aligned}
Cov(u(r_1, c, g, \varphi_0, \varphi_1), u(r_2, c, g, \varphi_0, \varphi_1)) \approx v(r_1, r_2) + \\
+\mathrm{E}[\mathrm{cu}(\mathrm{r}_1, 0, \mathrm{g}, \varphi_0, \varphi_1)\mathrm{u}^{(1)}(\mathrm{r}_2, 0, \mathrm{g}, \varphi_0, \varphi_1) + \\
+cu(r_2, 0, g, \varphi_0, \varphi_1) + u^{(1)}(r_1, 0, g, \varphi_0, \varphi_1)] + \\
\mathrm{E}\left[\mathrm{c}^2\mathrm{u}^{(1)}(\mathrm{r}_1, 0, \mathrm{g}, \varphi_0, \varphi_1)\mathrm{u}^{(1)}(\mathrm{r}_2, 0, \mathrm{g}, \varphi_0, \varphi_1)\right].
\end{aligned}
$$

Numerical Results

Let us consider the Dirichlet problem for the Helmholtz equation in a domain $0 \le x, y, z \le 1$:

$$\Delta u + cu = 0,$$
$$u|_\Gamma = \cos\left(x\sqrt{c/3}\right)\cos\left(y\sqrt{c/3}\right)\cos\left(z\sqrt{c/3}\right).$$

The exact solution is
$u(x, y, z; c) = \cos\left(x\sqrt{c/3}\right)\cos\left(y\sqrt{c/3}\right)\cos\left(z\sqrt{c/3}\right).$
For this problem $c^* = 3\pi^2 \approx 29.609.$

This problem solved in two ways. First the solution computed by standard random walks on spheres estimate

$$\eta_\varepsilon = u(r_N, c) \prod_{j=0}^{N-1} \frac{\sqrt{c} d_j}{\sin(\sqrt{c} d_j)}$$

at the point $r = (x, y, z)$ where $x = y = z = 0,9$. The numerical results are presented in Table 1. Second the solution computed by probabilistic representation (??). Using Euler scheme with a constant time step $\triangle t$, approximations r_i of the trajectory for the random process ξ_t at the points of time $i \triangle t$ was modeled statically. The solution computed by the following estimates

$$\zeta_1 = e^{cT} \varphi(r_N), \quad \zeta_2 = \left[\sum_{i=0}^{M} \frac{c_0^i T^i}{i!} \right] e^{(c-c_0)T} \varphi(r_N).$$

Here T is the approximate moment of the first exit of the process r_i from on a boundary of the cub, r_N is approximate coordinate of an exit. The corresponding numerical results are presented in Table 2.

Table 1.

c	$N \times 10^{-6}$	ε	$-u(r)$	$-\tilde{u}(r)$	$\lvert u(r) - \tilde{u}(r)\rvert \pm \sqrt{\frac{D\eta}{N}}$
50	16,7	10^{-4}	0.639	280.4	281 ± 5674
50	1	10^{-4}	0.639	131162	131162 ± 131464
35	16.7	10^{-4}	0.993	1.493	0.5 ± 0.260
35	16.7	10^{-2}	0.993	0.337	0.656 ± 0.918
30	9.8	10^{-4}	0.875	0.899	0.023 ± 0.018
20	1	10^{-4}	0.3197	0.3252	0.0055 ± 0.004
15	0.4	10^{-4}	0.078	0.0785	0.0004 ± 0.0003

Table 2.

c_0	c	$N \times 10^6$	$\triangle t{*}10^{-2}$	$-u(r)$	$-\widetilde{u}(r)$	$\lvert u(r) - \widetilde{u}(r)\rvert \pm$
–	30	10	1	0.875	0.790	0.085 ± 0.227
–	35	1	1	0.993	5.385	4.392 ± 3.452
5	30	1	1	0.875	1.143	0.268 ± 0.185
5	30	10^2	1	0.875	1.086	0.211 ± 0.179
10	35	10^2	10^{-2}	0.993	2.923	1.93 ± 0.516

If the solution calculated by estimate ζ_1 then we put $'-'$ in the first column of Table 2. We assume that $M = 50$ when solution calculated by estimate ζ_2.

References I

- G. A. Mikhailov
  Weight Algorithms in Statistical Modeling.
  (Nauka, Novosibirsk, 2003) [in Russian].
- S. M. Ermakov and G. A. Mikhailov
  Statistical Modeling.
  (Nauka, Moscow, 1982) [in Russian].
- Sabelfeld K.K.
  Monte Carlo Methods for boundary problems..
  Moskow.: Nauka, 1989[Engl. transl.: Springer–Verlag, 1991].

References II

Elepov B.S., Mikhailov G.A.
About solution to the Dirihlet problem for equation $(\Delta + c)u = -g$ by constructed 'walk by spheres'.
J. Num. Math. and Math. Physics. – 1969. – No 3. – P. 647–654.

Kublanovskaya V.N.
Using the analytical extention by transformation of variable.
Works of Steklov Math. Inst. 1959, v. 53, p. 145-185 .

Lukinov V.L., Mikhailov G.A.
The probabilistic representation and Monte Carlo methods for the first boundary value problem for a polyharmonic equation.
Rus. J. Num. Anal. and Math. Modell. 2004, v. 19, N.5, p. 434-449 .

References III

M. E. Muller
Some Continuous Monte Carlo Methods for the Dirichlet Problem.
Ann. Math. Stat. 27 (3), 569–589 (1956).