Monte Carlo Methods Based on Analytic Extension of the Resolvent of the Helmholtz Equation\textsuperscript{1}

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1 Helmholtz equation
   Monte Carlo approach
   Probabilistic representation for the solution

2 Problems
   Neumann series
   Eigenvalue algorithm

3 Analytical extension

4 Theoretical Results
   Walking on spheres
   The analytical extension of the probabilistic representation
   Covariance function of the biharmonic equation

5 Numerical Results
Let us consider the Dirichlet problem for the Helmholtz equation in a domain \( D \subset \mathbb{R}^3 \) with boundary \( \Gamma \):

\[
(\Delta + c)u = -g, \quad u|_{\Gamma} = \varphi
\]

(1)

Let us assume that the following conditions hold. The function \( g \) is satisfied Holder condition in \( \overline{D} \), \( D \) is a bounded open set in \( \mathbb{R}^3 \) with a regular boundary \( \Gamma \), the function \( \varphi \) is continuous on \( \Gamma \), \( c < c^* \), where \(-c^*\) is the first eigen value of Laplace operator defined on the domain \( D \).
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The solution as Neumann series

It is well known that solution to the problem (1) satisfies the integral equation

\[ u = \int_D k(r, r'; c)u(r')dr' + h(r), \text{ or } u = K_cu + h, \quad (2) \]

One of approaches to constructing statistical algorithms for solving (2) is based on the following presentation

\[ u = (I - K_c)^{-1} h = h + R_ch, \quad R_c = \sum_{i=1}^{\infty} K_c^i \quad (3) \]
Eigenvalue algorithm

Let us consider the following problem

\[ Ku = \lambda u \]

We can use well known formula to calculate first eigenvalue

\[ \lambda_1 = \lim_{i \to \infty} \frac{a_i}{a_{i+1}} \]

\[ R_c h = \sum_{i=0}^{\infty} a_i c^i, \quad a_i = K^{i+1} h \]
Using the same methods by parametric derivatives. We consider the parametric derivatives $u^{(m)}$ of the solution to the particular problem [Mikhailov G.A.]

$$(\Delta + c)u = 0, \quad u|_\Gamma = 1$$

It obviously, that

$$(\Delta + c)u^{(1)} = -u, \quad u^{(1)}|_\Gamma = 0$$

$$(\Delta + c)u^{(m)} = -mu^{(m-1)}, \quad u^{(m)}|_\Gamma = 0, \quad m = 1, 2, \ldots$$

Therefore

$$c^* - c = \lim_{m \to \infty} \frac{mu^{(m-1)}}{u^{(m)}}$$
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**Outlook**
Probabilistic representation

It is known (see [Elepov B.S., Mikhailov G.A. (1969)]), that under above stated conditions the probability representation for the solution to problem (1) has the form

\[ u(x) = \mathbb{E}\left[ \int_0^\tau e^{s(t;c)} g(\xi_t) \, dt + e^{s(\tau;c)} \varphi(\xi_\tau) \right], \quad s(t; c) = \int_0^t c(\xi_{t'}) \, dt', \]

where \( \xi_t \) is the diffusion process corresponding to the Laplace operator and starting at the point \( x \), \( \tau \) is the moment of the first exit of the process from the domain \( D \).
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Numerical Results
The convergence of Neumann series

Let us consider the remainder of Neumann series

\[ \text{Rem}_n = \sum_{i=n+1}^{\infty} a_i c^i \]

When \( u(r,c) \) is meromorphic function we have the following estimate [Kublanovskaya V.N., 1959]

\[ |\text{Rem}_n| = O(\delta^{n+1} n^{r-1}) \]

Here \( \delta = \left| c / c_1^{(k)} \right| \), \( c_1^{(k)} \) is any of the poles lying on the convergence circle, \( r \) is the highest multiplicity of this poles
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The relative error of the eigenvalue algorithm

Now note that

$$
\varepsilon_i = \left| \left( \frac{a_i}{a_{i+1}} - \lambda_1 \right) / \lambda_1 \right|
$$

In this case

$$
\varepsilon_i = O(\delta_1^{n+1}),
$$

where $\delta_1 = |\lambda_1 / \lambda_2|$, when $\lambda_1, \lambda_2$ are the simple poles.

$$
\varepsilon_i = O(\delta_1^{n+1} r^{-1}),
$$

when $\lambda_1$ is the simple pole, $\lambda_2$ is the multiple pole ( $r$ is the multiplicity of pole).
Map of spectral parameter

\[ \varphi : \eta \to \mathbf{c}, \quad \varphi(\eta) = \sum_{i=1}^{\infty} s_i \eta^i. \]

\[ F(\eta) = u(\varphi(\eta), r) = \sum_{i=0}^{\infty} b_i \eta^i, \]

where

\[ b_i = \sum_{k=0}^{i} a_k d_k^{(i)}, \quad d_k^{(i)} = \frac{1}{i!} \left( \frac{\partial^j}{\partial \eta^j} (\varphi(\eta))^k \right)_{\eta=0} \]
\[ u(c, r) = u(\varphi(\varphi^{-1}(c)), r) = F(\eta) = \sum_{i=0}^{\infty} b_i \eta^i = \]

\[ = \sum_{i=0}^{\infty} \eta^i \sum_{k=1}^{i} d_k^{(i)} a_i = \sum_{k=1}^{\infty} a_i l_i, \]

where \( l_i = \sum_{k=i}^{\infty} d_k^{(i)} \eta^i \)
\[ u(c, r) = \mathbb{E}\left[ \int_0^\tau e^{\varphi(\eta)t} g(\xi_t) dt + e^{\varphi(\eta)\tau} \phi(\xi_\tau) \right] \]

\[ e^{\varphi(\eta)t} = \sum_{i=0}^\infty \frac{\varphi^i(\eta)t^i}{i!} = \sum_{i=0}^\infty b_i \eta^i, \]

where \( b_i = \sum_{k=1}^i d_k^{(i)} \frac{t^k}{k!} \)

\[ u(c, r) \approx \sum_{k=0}^\infty \mathbb{E}\left[ \int_0^\tau \frac{t^k}{k!} \left( \sum_{n=k}^m d_k^{(n)} \eta^n \right) g(\xi_t) dt + e^{\varphi(\eta)\tau} \phi(\xi_\tau) \right] \]
Certain maps

- Kublanovskaya V.N., Sabelfeld K.K.
  \[ c = \varphi(\eta) = \frac{4\alpha\eta}{(1 - \eta)^2} \]

- Mikhailov G.A.
  \[ c = \varphi(\eta) = \frac{\alpha}{(1 - \beta\eta)} \]

- \[ c = \varphi(\eta) = \alpha + \beta \arctan \eta \]
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5. **Numerical Results**
Let us consider the following problem

\[
\begin{align*}
(\Delta + c)^{p+1}u &= -g, \\
(\Delta + c)^k u|_\Gamma &= \varphi_k, \quad k = 0, \ldots, p.
\end{align*}
\]
Metaharmonic equation


Under above stated conditions, the $p$-th parametric derivative of the solution to problem (1) with the functional parameters

$$
\varphi = \sum_{k=0}^{p} \frac{(-1)^k (c - c_1)^{p-k}}{p!} \varphi_k, \quad g_1 = \frac{(-1)^p}{p!} g
$$

(6)

is a solution to problem (1) (all derivatives are calculated at the point $c_1$). Here $p$-th parametric derivative satisfies the following problem

$$
\begin{align*}
(\Delta + c)^{p+1} u &= -g^i \frac{c_0^{i+1}}{i!}, \\
(\Delta + c)^k u|_\Gamma &= \varphi^i \frac{c_0^{i+1}}{i!}, \quad k = i - 1, \\
(\Delta + c)^k u|_\Gamma &= 0, \quad k = i - 2, \ldots, 0.
\end{align*}
$$

(7)
Theorem (2)

If $c < c^*$ and the first spatial derivatives of the functions $\{u_k^{(i)} \}$, $i = 1, \ldots, p + 1$, are uniformly bounded in $\bar{D}$, then

$$|u(r) - \mathbb{E}\tilde{\eta}_{1,\varepsilon}^{(p)}| \leq C_p\varepsilon, \quad r \in D, \quad \varepsilon > 0,$$

where

$$\tilde{\eta}_{1,\varepsilon}^{(p)} = \sum_{i=0}^{N} \left\{ \left[ \prod_{j=0}^{i-1} s(c, d_j) \right] g(\rho_i) \frac{d_i^2 G(\rho; c, d_i)}{2nG(\rho; 0, d_i)} \right\}^{(p)} (8)$$

$$+ \left\{ \left[ \prod_{j=0}^{N-1} s(c, d_j) \right] \varphi(r_N, c) \right\}^{(p)},$$
Metaharmonic equation

Here, $d_j = d(r_j)$, $D(r_i)$ is a ball of radius $d_i$ with the center at the point $r_i$, $G(\rho; c, d)$ is the spherical Green’s function; and

$$s(c, d) = \begin{cases} 
    d \sqrt{c} / \sin (d \sqrt{c}), & c \geq 0; \\
    d \sqrt{|c|} / \sinh (d \sqrt{|c|}), & c \leq 0.
\end{cases}$$
The practical estimates

\[ \tilde{z}(\rho) = \sum_{i=0}^{N} \left\{ \prod_{j=0}^{i-1} \frac{\sqrt{cd_j}}{\sin(\sqrt{cd_j})} \right\} g(\nu_i, \omega_i) \frac{d_i^3 \sqrt{c} \sin(\sqrt{c}(d_i - \nu_i))}{6(d_i - \nu_i) \sin(\sqrt{cd_i})} \]

\[ + \left\{ \prod_{j=0}^{N-1} \frac{\sqrt{cd_j}}{\sin(\sqrt{cd_j})} \right\} \varphi(r_N, c) \]
The practical estimates

\[ \eta_k = \sum_{p=0}^{k} \left[ \sum_{i=0}^{N} \left\{ \prod_{j=0}^{i-1} \frac{\sqrt{cd_j}}{\sin(\sqrt{cd_j})} \right\} \frac{c_0^{i+1}}{i!} g(\nu_i, \omega_i) \frac{d_i^3 \sqrt{c} \sin(\sqrt{c}(a - d_i))^3}{6(d_i - \nu_i) \sin(\sqrt{c}(a - d_i))} \right] \\
+ \left\{ \prod_{j=0}^{N-1} \frac{\sqrt{cd_j}}{\sin(\sqrt{cd_j})} \left[ \frac{c_0^{i+1}}{i!} \varphi(r_N, c) \right] \right\}^{(p)} \]

The random variables \( \nu_i \) distributed in the interval \((0, d_i)\) with the probability density \( 6x(1 - x/d_i)d_i^{-2} \) and the unit isotropic vectors \( \omega_i \) are modeled by means of the well-known formulas [S. M. Ermakov and G. A. Mikhailov, 1982].
Outlook

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Lemma
Under above stated conditions, and $c < c^*$. Then

$$
\frac{\partial^p u}{\partial c^p} = u^{(p)} = E\left[ \int_0^\tau t^p e^{ct} g(\xi_t) dt + \sum_{l=0}^p C_p^l \tau^{p-l} e^{ct} \frac{\partial^l}{\partial c^l} \varphi(\xi_\tau, c) \right].
$$
On the other hand we have

$$u(c, r) = \mathbb{E} \left[ \int_0^\tau e^{ct} g(\xi_t) dt + e^{c\tau} \varphi(\xi_\tau) \right] =$$

$$= \sum_{i=0}^{\infty} \mathbb{E} \left[ \int_0^\tau \frac{c^i t^i}{i!} g(\xi_t) dt + \frac{c^i \tau^i}{i!} \varphi(\xi_\tau) \right] = \sum_{i=0}^{\infty} c^i a_i,$$

where $a_i = \mathbb{E} \left[ \int_0^\tau \frac{t^i}{i!} g(\xi_t) dt + \frac{\tau^i}{i!} \varphi(\xi_\tau) \right]$;

$$R_n = \sum_{i=n}^{\infty} c^i a_i = O(\left| c/c^* \right|^{n+1} n^{r-1}).$$
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The oscillations of the plate in a bounded region $D \subset \mathbb{R}^2$ under action of the random field stress $\sigma(r) = -g(r)$ describes equation of the form

$$\Delta^2 u + cu = -g, \quad u|_\Gamma = \varphi_0, \quad \Delta u|_\Gamma = \varphi_1,$$

(9)

thus it is possible consider the randomness of the boundary functions $\varphi_0(r)$ and $\varphi_1(r)$ [Bolotin V.V., 1979] also.

The solution to this problem is a random field. Required to determine its covariance function $v(r, r') = \mathbb{E}[u(r)u(r')]$.

It is assumed that $\mathbb{E}g(r) \equiv \mathbb{E}\varphi_0(r) \equiv \mathbb{E}\varphi_1(r) \equiv 0$ and, hence, $\mathbb{E}u(r) \equiv 0$. 

Covariance function
Solution \( u(r, c, g, \varphi_0, \varphi_1) \) can be approximately replaced the corresponding partial sum of Loran series. To construct estimates of the unknown parameter derivatives of \( u^{(1)}(r, 0, g, \varphi_0, \varphi_1) \), \( u^{(2)}(r, 0, g, \varphi_0, \varphi_1) \) we use the following way. Differentiating with respect to the parameter \( c \) equation (??), we get for \( c = 0 \):

\[
\Delta^2 u^{(1)} = -u, \quad u^{(1)}\big|_{\Gamma} = \Delta u^{(1)}\big|_{\Gamma} = 0. \tag{10}
\]
Hence it is clear that the parametric derivative $u^{(1)}(r, 0, g, \varphi_0, \varphi_1)$ is the solution of the following problem:

$$\begin{align*}
\Delta^4 u &= -g, \quad u\big|_{\Gamma} = 0, \quad \Delta u\big|_{\Gamma} = 0, \\
\Delta^2 u\big|_{\Gamma} &= -\varphi_0, \quad \Delta^3 u\big|_{\Gamma} = -\varphi_1.
\end{align*}$$

(11)

The parametric derivative $u^{(2)}(r, 0, g, \varphi_0, \varphi_1)/2$ is the solution of the following problem:

$$\begin{align*}
\Delta^6 u &= g, \quad \Delta^k u\big|_{\Gamma} = 0, \quad k = 0, \ldots, 3, \\
\Delta^4 u\big|_{\Gamma} &= \varphi_0, \quad \Delta^5 u\big|_{\Gamma} = \varphi_1
\end{align*}$$

(12)
Estimates for solutions of (13), (14) respectively are as follows:

$$\tilde{\eta}_{i,\varepsilon}^{(3)} = \sum_{i=0}^{N} \left\{ \sum_{k=0}^{3} C_{3}^{k} S_{i}^{(3-k)}(0) \frac{G^{(k)}(\rho; 0, d_{i})}{G(\rho; 0, d_{i})} \right\} \frac{[-d_{i}^{2} g(\rho_{i})]}{24},$$

$$+ S_{N}^{(3)}(0) \frac{\varphi_{1}(r_{N})}{6} - S_{N}^{(2)}(0) \frac{\varphi_{0}(r_{N})}{2},$$

$$\tilde{\eta}_{i,\varepsilon}^{(5)} = \left\{ \sum_{i=0}^{N} \sum_{k=0}^{5} C_{5}^{k} S_{i}^{(5-k)}(0) \frac{G^{(k)}(\rho; 0, d_{i})}{G(\rho; 0, d_{i})} \right\} \frac{d_{i}^{2} g(\rho_{i})}{480},$$

$$- S_{N}^{(5)}(0) \frac{\varphi_{1}(r_{N})}{120} + S_{N}^{(4)}(0) \frac{\varphi_{0}(r_{N})}{24},$$

where $$S_{i}(c) = \prod_{j=0}^{i-1} s(c, d_{j}).$$
Representation of estimates shows that it is necessary to find values derivatives of $S_i^{(k)}(0)$ for $k = 0, 3$. Differentiating with respect to parameter $c$ the logarithmic derivative of $S^{(1)} = S(\ln S)^{(1)}$, we can easily obtain the following relations

\[
S^{(2)} = S^{(1)}(\ln S)^{(1)} + S(\ln S)^{(2)},
\]

\[
S^{(3)} = S^{(2)}\ln S^{(1)} + 2S^{(1)}(\ln S)^{(2)} + S(\ln S)^{(3)},
\]

\[
S^{(4)} = S^{(3)}\ln S^{(1)} + 3S^{(2)}(\ln S)^{(2)} + 3S^{(1)}(\ln S)^{(3)} + S(\ln S)^{(4)},
\]

\[
S^{(5)} = S^{(4)}\ln S^{(1)} + 4S^{(3)}(\ln S)^{(2)} + 6S^{(2)}(\ln S)^{(3)} + 4S^{(1)}(\ln S)^{(4)} + S(\ln S)^{(5)}.
\]
Derivatives of \((\ln S)^{(k)}\), \(k = 1, 5\) at \(c = 0\) is easily obtained using equations

\[
s(c, d) = 1 + \frac{d^2}{4} c + \frac{3d^4}{2^6} c^2 + \frac{19d^6}{2^83^2} c^3 + \frac{211}{2^143^2} c^4 + \frac{1217}{2^16 \cdot 3} c^5.
\]

\[
(\ln S_i)^{(1)} = \sum_{j=0}^{i-1} \frac{s^{(1)}(0, d_j)}{s(0, d_j)} = \sum_{j=0}^{i-1} \frac{d_j^2}{4},
\]

\[
(\ln S_i)^{(2)} = \sum_{j=0}^{i-1} \left\{ \frac{s^{(2)}(0, d_j)}{s(0, d_j)} - \left( \frac{s^{(1)}(0, d_j)}{s(0, d_j)} \right)^2 \right\} = \sum_{j=0}^{i-1} \frac{d_j^4}{2^5},
\]

\[
(\ln S_i)^{(3)} = \sum_{j=0}^{i-1} \left\{ \frac{s^{(3)}(0, d_j)s(0, d_j) - s^{(2)}(0, d_j)s^{(1)}(0, d_j)}{s^2(0, d_j)} \right\}
- \frac{2s^{(1)}(0, d_j)[s^{(2)}(0, d_j)s(0, d_j) - (s^{(1)}(0, d_j))^2]}{s^3(0, d_j)} = \sum_{j=0}^{i-1} \frac{d_j^6}{2^7},
\]

\[
(\ln S_i)^{(4)} = \sum_{j=0}^{i-1} \left\{ \frac{s^{(4)}(0, d_j)}{s(0, d_j)} - \frac{4s^{(3)}(0, d_j)s^{(1)}(0, d_j)}{s^2(0, d_j)} \right\}
+ \frac{4s^{(2)}(0, d_j)[s^{(3)}(0, d_j)s^{(1)}(0, d_j) - (s^{(1)}(0, d_j))^2]}{s^4(0, d_j)} = \sum_{j=0}^{i-1} \frac{d_j^8}{2^9}.
\]
Using the following asymptotic relations

\[
\frac{J_0(z\sqrt{c})}{J_0(d\sqrt{c})} \sim 1 + \frac{d^2 - z^2}{4} c + \frac{3d^4 - 4z^2d^2 + z^4}{2^6} c^2 \\
+ \frac{19d^6 - 3^3z^2d^4 + 3^2z^4d^2 - z^6}{2^83^2} c^3 \\
+ \frac{211d^8 - 19 \cdot 2^4d^6z^2 + 3^32^2d^4z^4 - 2^4d^2z^6 + z^8}{2^{14}3^2} c^4 \\
+ \left[ \frac{3 \cdot 1217d^{10} - 211 \cdot 5^2d^8z^2 + 19 \cdot 2^2 \cdot 5^2d^6z^4}{2^{16} \cdot 3^2 \cdot 5^2} \\
+ \frac{-2^2 \cdot 3 \cdot 5^2d^4z^6 + 5^2d^2z^8 - z^{10}}{2^{16} \cdot 3^2 \cdot 5^2} \right] c^5
\]

\[
N_0^{(2)}(z\sqrt{c}) \sim \frac{2}{\pi} \left[ -\frac{1}{2c^2} - \frac{z^2}{2^3c} - \frac{3z^4}{2^7} + \left( \gamma + \ln \frac{z\sqrt{c}}{2} \right) \frac{z^4}{2^5} \right],
\]

\[
N_0^{(3)}(z\sqrt{c}) \sim \frac{2}{\pi} \left[ \frac{1}{c^3} + \frac{z^2}{2^3c^2} + \frac{z^4}{2^6c} + \frac{z^6}{2^{8}3^2} - \left( \gamma + \ln \frac{z\sqrt{c}}{2} \right) \frac{z^6}{2^{14}3^2} \right],
\]

\[
N_0^{(4)}(z\sqrt{c}) \sim \frac{2}{\pi} \left[ -\frac{3}{c^4} - \frac{z^2}{4c^3} - \frac{z^4}{2^6c^2} - \frac{z^6}{2^{8}3c} - \frac{25z^8}{2^{14}3^2} + \left( \gamma + \ln \frac{z\sqrt{c}}{2} \right) \frac{z^8}{2^{27}3} \right].
\]
we derive that

\[ G^{(2)}(\rho; 0, d) = \frac{1}{2^8 \pi} (6z^4 - 16z^2d^2 + 10d^4 + 4z^4 \ln \frac{d}{z}), \]

\[ G^{(3)}(\rho; 0, d) = \frac{1}{2^{10} 3^2 \pi} (175d^6 - 270d^4z^2 + 108d^2z^4 - 13z^6 - 1), \]

\[ G^{(4)}(\rho; 0, d) = \frac{1}{2\pi} \left( \frac{677d^8}{2^{13} 3} + \frac{25z^8}{2^{13} 3^2} - \frac{z^6d^2}{2^{7} 3} - \frac{23d^6z^2}{2^{6} 3^2} + \frac{15z^4d^4}{2^{10}} \right), \]

\[ G^{(5)}(\rho; 0, d) = \frac{1}{2\pi} \left( \frac{71 \cdot 103d^{10}}{2^{11} \cdot 3 \cdot 5^2} - \frac{137z^{10}}{2^{15} \cdot 3^2 \cdot 5^2} + \frac{5z^8d^2}{2^{13} 3} - \frac{5 \cdot 67z^6d^4}{2^{12} 3} + \frac{5 \cdot 23z^4d^6}{2^{10} 3^2} - \frac{z^{10}}{2^{13} \cdot 3 \cdot 5 \ln \frac{d}{z}} \right). \]
\[
\text{Cov}(u(r_1, c, g, \varphi_0, \varphi_1), u(r_2, c, g, \varphi_0, \varphi_1)) \approx v(r_1, r_2) + \\
\quad + E[cu(r_1, 0, g, \varphi_0, \varphi_1)u^{(1)}(r_2, 0, g, \varphi_0, \varphi_1) + \\
\quad \quad + cu(r_2, 0, g, \varphi_0, \varphi_1) + u^{(1)}(r_1, 0, g, \varphi_0, \varphi_1)] + \\
\quad E \left[ c^2 u^{(1)}(r_1, 0, g, \varphi_0, \varphi_1)u^{(1)}(r_2, 0, g, \varphi_0, \varphi_1) \right].
\]
Numerical Results

Let us consider the Dirichlet problem for the Helmholtz equation in a domain $0 \leq x, y, z \leq 1$:

$$\Delta u + cu = 0,$$

$$u|_\Gamma = \cos\left(x\sqrt{c/3}\right)\cos\left(y\sqrt{c/3}\right)\cos\left(z\sqrt{c/3}\right).$$

The exact solution is

$$u(x, y, z; c) = \cos\left(x\sqrt{c/3}\right)\cos\left(y\sqrt{c/3}\right)\cos\left(z\sqrt{c/3}\right).$$

For this problem $c^* = 3\pi^2 \approx 29.609$. 
This problem solved in two ways. First the solution computed by standard random walks on spheres estimate

\[ \eta_\varepsilon = u(r_N, c) \prod_{j=0}^{N-1} \frac{\sqrt{c}d_j}{\sin(\sqrt{c}d_j)} \]

at the point \( r = (x, y, z) \) where \( x = y = z = 0,9 \). The numerical results are presented in Table 1. Second the solution computed by probabilistic representation (??). Using Euler scheme with a constant time step \( \Delta t \), approximations \( r_i \) of the trajectory for the random process \( \xi_t \) at the points of time \( i\Delta t \) was modeled statically. The solution computed by the following estimates

\[ \zeta_1 = e^{cT}\varphi(r_N), \quad \zeta_2 = \left[ \sum_{i=0}^{M} \frac{c_i^i T^i}{i!} \right] e^{(c-c_0)T}\varphi(r_N). \]
Here $T$ is the approximate moment of the first exit of the process $r_i$ from on a boundary of the cub, $r_N$ is approximate coordinate of an exit. The corresponding numerical results are presented in Table 2.
| $c$ | $N \times 10^{-6}$ | $\varepsilon$ | $-u(r)$ | $-\tilde{u}(r)$ | $|u(r) - \tilde{u}(r)| \pm \sqrt{\frac{D_\eta}{N}}$ |
|-----|-------------------|----------|---------|---------------|----------------------------------|
| 50  | 16.7              | $10^{-4}$| 0.639   | 280.4         | $281 \pm 5674$                  |
| 50  | 1                 | $10^{-4}$| 0.639   | 131162        | $131162 \pm 131464$             |
| 35  | 16.7              | $10^{-4}$| 0.993   | 1.493         | $0.5 \pm 0.260$                 |
| 35  | 16.7              | $10^{-2}$| 0.993   | 0.337         | $0.656 \pm 0.918$               |
| 30  | 9.8               | $10^{-4}$| 0.875   | 0.899         | $0.023 \pm 0.018$               |
| 20  | 1                 | $10^{-4}$| 0.3197  | 0.3252        | $0.0055 \pm 0.004$              |
| 15  | 0.4               | $10^{-4}$| 0.078   | 0.0785        | $0.0004 \pm 0.0003$             |

Table 1.
If the solution calculated by estimate $\zeta_1$ then we put ‘−’ in the first column of Table 2. We assume that $M = 50$ when solution calculated by estimate $\zeta_2$.

| $c_0$ | $c$  | $N \times 10^6$ | $\triangle t \times 10^{-2}$ | $-u(r)$ | $-\tilde{u}(r)$ | $|u(r) - \tilde{u}(r)| \pm$  |
|-------|------|----------------|----------------------------|---------|----------------|--------------------------|
| −     | 30   | 10             | 1                          | 0.875   | 0.790          | 0.085 ± 0.227            |
| −     | 35   | 1              | 1                          | 0.993   | 5.385          | 4.392 ± 3.452            |
| 5     | 30   | 1              | 1                          | 0.875   | 1.143          | 0.268 ± 0.185            |
| 5     | 30   | $10^2$         | 1                          | 0.875   | 1.086          | 0.211 ± 0.179            |
| 10    | 35   | $10^2$         | $10^{-2}$                  | 0.993   | 2.923          | 1.93 ± 0.516             |
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