Optimal Approximation Of The Solutions Of The Stochastic Differential Equations With Irregular Coefficients

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Problem formulation:

$T > 0$, $\sigma_1, \sigma_2 : [0, T] \to \mathbb{R}$, $a, b : \mathbb{R} \to \mathbb{R}$,
$\{W(t)\}_{t \in [0,T]}$ – one–dimensional Brownian motion,
$\mathbb{E}\eta^2 < +\infty$, η – independent of W,

$$\begin{cases} dX(t) = \sigma_1(t)a(X(t))dt + \sigma_2(t)b(X(t))dW(t), & t \in [0, T], \\ X(0) = \eta, \end{cases} \tag{1}$$

σ_1, σ_2 – regular or singular, a, b – at least Lipschitz functions,

Aim

Possible optimal approximation of $X(T)$.

Classes of coefficients:

$\sigma_1, \sigma_2 \in \mathcal{F}^{\varrho}_{\mathrm{sng},p}$, $\varrho \in (0,1]$, $p \in \mathbb{N}_+ \cup \{0\}$,

- $\mathcal{F}^{\varrho}_{\mathrm{sng},p}$, $p \geqslant 0$ – class of piecewise Hölder continuous functions

$$\Delta_g^k := g(s_g^{k+}) - g(s_g^{k-}), \quad k = 1, \ldots, q = q(g),\ 1 \leqslant q \leqslant p,$$

$a, b \in \mathcal{D}^1 \vee \mathcal{D}^2$,

- $\mathcal{D}^1 \subset C(\mathbb{R})$: $f-$ bounded Lipschitz functions,
- $\mathcal{D}^2 \subset \mathcal{D}^1 \cap C^1(\mathbb{R})$: f'–absolutely continuous in $\mathbb{R}$.

Classes of input data $(\sigma_1, \sigma_2, a, b, \eta)$:

(F1) $F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2} =$
$\mathcal{F}^{\varrho}_{\mathrm{sng},p_1} \times \mathcal{F}^{\varrho}_{\mathrm{sng},p_2} \times \mathcal{D}^2 \times \{f \equiv \ const\} \times \{\eta \mid \mathbb{E}\eta^2 \leqslant \bar{L}\}$,

(F2) $F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2} = \mathcal{F}^{\varrho}_{\mathrm{sng},p_1} \times \mathcal{F}^{\varrho}_{\mathrm{sng},p_2} \times \mathcal{D}^1 \times \mathcal{D}^1 \times \{\eta \mid \mathbb{E}\eta^2 \leqslant \bar{L}\}$,

$\varrho \in (0,1]$, $p_1, p_2 \in \mathbb{N}_0$, $\bar{L} > 0$,

$p_1 = p_2 = 0 \Rightarrow$ Regular case,

$F^{\mathrm{add},\varrho}_{\mathrm{sng},0,0} \subset F^{\mathrm{mul},\varrho}_{\mathrm{sng},0,0} \subset F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2}$,
$F^{\mathrm{add},\varrho}_{\mathrm{sng},0,0} \subset F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2} \subset F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2}$, $p_1 + p_2 \geqslant 1$

Algorithm:

$$\mathcal{A}(\sigma_1, \sigma_2, a, b, \eta, W) = \psi(N(\sigma_1, \sigma_2, a, b, \eta, W)), \quad \psi : \mathbb{R}^n \to \mathbb{R},$$

Ψ_n– adapt (a, b), Ψ_n^*–adapt $(a, b, \sigma_1, \sigma_2)$, $\Psi_n \subset \Psi_n^*$,

Model of error:

$$e(\mathcal{A}, (\sigma_1, \sigma_2, a, b, \eta)) = \left(\mathbb{E}\Big|X(T) - \mathcal{A}\Big(\sigma_1, \sigma_2, a, b, \eta, W\Big)\Big|^2\right)^{1/2}, \tag{2}$$

$$e(\mathcal{A}, \mathcal{G}) = \sup_{(\sigma_1, \sigma_2, a, b, \eta) \in \mathcal{G}} e(\mathcal{A}, (\sigma_1, \sigma_2, a, b, \eta)), \tag{3}$$

$$e_n(\mathcal{G}) = \inf_{\mathcal{A} \in \Psi_n} e(\mathcal{A}, \mathcal{G}), \quad e_n^*(\mathcal{G}) = \inf_{\mathcal{A} \in \Psi_n^*} (\mathcal{A}, \mathcal{G}), \tag{4}$$

$$e_n^*(\mathcal{G}) \leqslant e_n(\mathcal{G}).$$

Known results:

- $\Big(\sigma_2 \equiv 0 \vee b \equiv 0\Big) \wedge a \equiv const \wedge \sigma_1$-singular $\Rightarrow$

$$X(T) = \int\limits_0^T \sigma_1(t)dt, \tag{5}$$

Plaskota, Wasilkowski (2005),

- $\Big(\sigma_2 \equiv 0 \vee b \equiv 0\Big) \wedge \sigma_1 \equiv const \wedge a$-singular $\Rightarrow$

$$dX(t) = a(X(t))dt, \tag{6}$$

Kacewicz, Przybyłowicz (2008),

Known results:

- $\left(\sigma_1 \equiv 0 \vee a \equiv 0\right) \wedge b \equiv const \wedge \sigma_2$-singular $\Rightarrow$

$$X(T) = \int_0^T \sigma_2(t)dW(t), \tag{7}$$

  Woźniakowski, Wasilkowski (2001), Przybyłowicz (2009, 2010),
- Pointwise approximation of (1) with regular coefficients (asymptotic case) – Müller–Gronbach (2004).
- Weak convergence of the Euler scheme for SDEs with discontinuous coefficients – Yan (2002).

$n \in \mathbb{N}_+$, $0 = t_0 < t_1 < \ldots < t_n = T$,
$\Delta t_i = t_{i+1} - t_i$, $\Delta t_{\max} = \max\limits_{0 \leqslant i \leqslant n-1} \Delta t_i$,
$\Delta W_i = W(t_{i+1}) - W(t_i)$,

Euler algorithm X^E (Kloeden, Platen, 1992):

$$\begin{cases} \hat{X}^E(0) = \eta, \\ \hat{X}^E(t_{i+1}) = \hat{X}^E(t_i) + \sigma_1(t_i)a(\hat{X}^E(t_i))\Delta t_i + \sigma_2(t_i)b(\hat{X}^E(t_i))\Delta W_i, \end{cases}$$

$i = 0, 1, \ldots, n-1$,

$$X^E(\sigma_1, \sigma_2, a, b, \eta, W) = \hat{X}^E(T). \tag{8}$$

$X^E \in \Psi_{c_1 n}$, $c_1 > 0$

$g \in \mathcal{F}^{\varrho}_{\mathrm{sng},p}$,

$$m(g,i) = \sharp\Big(\{s_g^1, s_g^2, \dots, s_g^p\} \cap (t_i, t_{i+1})\Big), \ \ i = 0, 1, \dots, n-1,$$
$$\mathcal{N}(g) = \{i \mid 0 \leqslant i \leqslant n-1, m(g,i) > 0\},$$

For $i \in \mathcal{N}(g)$,
$\delta_i(g) =$ sum of absolute values of jumps of g at singular points $s_g^k \in (t_i, t_{i+1})$

Theorem 1. (Error of the Euler algorithm X^E)

For all $(\sigma_1, \sigma_2, a, b, \eta) \in F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2}$ *we have*

$$
\begin{aligned}
e(X^E, (\sigma_1, \sigma_2, a, b, \eta)) \leqslant \; & C_1 \Big(1 + \sum_{i \in \mathcal{N}(\sigma_1)} \delta_i(\sigma_1) \Big) (\Delta t_{\max})^{\varrho} \\
& + C_2 \sum_{i \in \mathcal{N}(\sigma_2)} \Big(\delta_i(\sigma_2) \cdot (\Delta t_i)^{1/2} \Big),
\end{aligned}
$$

while for all $(\sigma_1, \sigma_2, a, b, \eta) \in F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2}$ *we have*

$$
\begin{aligned}
e(X^E, (\sigma_1, \sigma_2, a, b, \eta)) \leqslant \; & K_1 \Big(1 + \sum_{i \in \mathcal{N}(\sigma_1)} \delta_i(\sigma_1) \Big) (\Delta t_{\max})^{\varrho} \\
& + K_2 \Big(1 + \sum_{i \in \mathcal{N}(\sigma_2)} \delta_i(\sigma_2) \Big) (\Delta t_{\max})^{1/2}.
\end{aligned}
$$

Corollary 1.

For the algorithm X^E based on the equidistant discretization with $\Delta t_{\mathrm{max}} = T/n$ and for $\varrho \in (0,1]$, $p_1 \geqslant 0$ we have

$$e(X^E, F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,0}) = O(n^{-\varrho}), \tag{9}$$

$$e(X^E, F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2}) = O(n^{-\min\{1/2,\varrho\}}) \quad \text{for} \quad p_2 \geqslant 1, \tag{10}$$

$$e(X^E, F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2}) = O(n^{-\min\{1/2,\varrho\}}) \quad \text{for} \quad p_2 \geqslant 0, \tag{11}$$

Results:

- $\varrho \in (0,1]$, $p_1 \geqslant 0, p_2 = 0$

$$e_n^*(F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,0}) = \Theta(n^{-\varrho}), \quad (12)$$

$$C_1 n^{-\varrho} \leqslant e_n^*(F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,0}) \leqslant C_2 n^{-\min\{1/2,\varrho\}}, \quad (13)$$

- $\varrho \in (0,1/2], p_1 \geqslant 0, p_2 \geqslant 1$

$$e_n^*(F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2}) = \Theta(n^{-\varrho}), \quad (14)$$

$$e_n^*(F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2}) = \Theta(n^{-\varrho}), \quad (15)$$

- $\varrho \in (1/2,1], p_1 \geqslant 0, p_2 \geqslant 2$

$$e_n^*(F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2}) = \Theta(n^{-1/2}), \quad (16)$$

$$e_n^*(F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,p_2}) = \Theta(n^{-1/2}). \quad (17)$$

- $\varrho \in (1/2, 1], p_1 \geqslant 0, p_2 = 1$

$$e_n(F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,1}) = \Theta(n^{-1/2}), \tag{18}$$

$$e_n(F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,1}) = \Theta(n^{-1/2}), \tag{19}$$

$$e^*_n(F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,1}) = \Theta(n^{-\varrho}), \tag{20}$$

$$C_1 n^{-\varrho} \leqslant e^*_n(F^{\mathrm{mul},\varrho}_{\mathrm{sng},p_1,1}) \leqslant C_2 n^{-1/2}, \tag{21}$$

(20) $\Rightarrow$ Upper bounds $\Rightarrow$ Construction of the Euler–type algorithm with an adaptive grid:

(i) detection of the unknown singularity of σ_2,
(ii) suitable modification of the starting grid,
(iii) perform X^E on the new mesh.

- $\varrho \in (1/2, 1]$, $p_1 \geqslant 0$, $p_2 \geqslant 2$, $\theta > 0$,

$$F^{\mathrm{add},\varrho,\theta}_{\mathrm{sng},p_1,p_2} = \Big\{ \Big(\sigma_1, \sigma_2, a, b, \eta\Big) \in F^{\mathrm{add},\varrho}_{\mathrm{sng},p_1,p_2} \mid \sigma_2 \in \mathcal{F}^{\varrho,\theta}_{\mathrm{sng},p_2} \Big\},$$

$$\mathcal{F}^{\varrho,\theta}_{\mathrm{sng},p} = \mathcal{F}^{\varrho}_{\mathrm{sng},1} \cup \{ g \in \mathcal{F}^{\varrho}_{\mathrm{sng},p} \mid \min_{1 \leqslant k \leqslant p-1} |s_g^{k+1} - s_g^k| > \theta \},$$

$\Delta t_{\max} = T/n$, $n_0 = \lceil T/\theta \rceil$, $n \geqslant n_0 \Rightarrow$ separated singularities,

$$e^*_{M(n)}(F^{\mathrm{add},\varrho,\theta}_{\mathrm{sng},p_1,p_2}) = O(n^{-\varrho}), \qquad (22)$$

cost: $M(n) = O(n \log_2 n)$

Asymptotic setting.

$g \in \mathcal{C}^{\varrho}_{\mathrm{sng}}([0,T])$, $\varepsilon \in \mathbb{R}$, $u \in [0,T)$

$$g_{\varepsilon,u} = g + \varepsilon \mathbf{1}_{[0,u)}, \; g_{\varepsilon,T} = g + \varepsilon. \tag{23}$$

$X_{\varepsilon,u}(T)$ - the solution of (1) for $\sigma_2 = \sigma_{2,\varepsilon,u}$, $b = \mathbf{1}_{\mathbb{R}}$.

Proposition 1.

Consider a sequence of algorithms $\{\mathcal{A}_n\}_{n=1}^{\infty}$ which use nonadaptive information for σ_2. For all $a \in C_{\mathrm{Lip}}(\mathbb{R})$, $\sigma_1, \sigma_2 \in \mathcal{C}^{\varrho}_{\mathrm{sng}}([0,T])$, $\varepsilon \neq 0$ the set

$$A = \Big\{ u \in [0,T] \mid \lim_{n \to \infty} n^{1/2} \cdot ||X_{\varepsilon,u}(T) - \mathcal{A}_n(\sigma_1, \sigma_{2,\varepsilon,u}, a, \mathbf{1}_{\mathbb{R}}, \eta, W)||_{L^2(\Omega)} = 0 \Big\}$$

is of Lebesgue measure zero.

What next?

- General Itô-Taylor schemes.
- Order of strong convergence of the Euler algorithm for SDEs with discontinuous a and/or b.
- Levy/fBm noise.

K.S. Chan, O. Stramer, *Weak consistency of the Euler method for numerically solving stochastic differential equations with discontinuous coefficients*, Stoch. Proc. and their Appl. 76 (1998), 33–44.

N. Halidias, P. E. Kloeden, *A note on the Euler–Maruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient*, BIT Numer. Math. 48 (2008), 51–59.

N. Hofmann, T. Müller–Gronbach, K. Ritter, *Optimal approximation of stochastic differential equations by adaptive step-size control*, Math. Comp. 69 (2000), 1017–1034.

N. Hofmann, T. Müller–Gronbach, K. Ritter, *The Optimal Discretization of Stochastic Differential Equations*, J. Complexity 17 (2001), 117–153.

- P. E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag Berlin, Heidelber, 1992.
- G. N. Milstein, M. V. Tretyakov, Stochastic Numerics for Mathematical Physics, Scientific Computation, Springer–Verlag Berlin Heidelberg, 2004.
- T. Müller–Gronbach, *Optimal pointwise approximation of SDEs based on Brownian motion at discrete points*, Annals Appl. Prob. 14 (2004), 1605–1642.
- E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics, vol. 1349, New York, Springer–Verlag, 1988.

- L. Plaskota, G. W. Wasilkowski, *Adaption allows efficient integration of functions with unknown singularities*, Numer. Math. 102 (2005), 123–144.
- L. Plaskota, G.W. Wasilkowski, *The power of adaptive algorithms for functions with singularities*, J. Fixed Point Theory Appl. 6 (2009), 227–248.
- L. Plaskota, G. W. Wasilkowski, Y. Zhao, *The power of adaption for approximating functions with singularities*, Math. Comp. 77 (2008), 2309–2338.
- B. Kacewicz, P. Przybyłowicz, *Optimal adaptive solution of initial–value problems with unknown singularities*, J. Complexity 24 (2008), 455–476.

P. Przybyłowicz, *Adaptive Itô–Taylor algorithm can optimally approximate the Itô integrals of singular functions*, J. Comp. and Appl. Math. 235 (2010), 203–217.

J. F. Traub, G.W.Wasilkowski, H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.

L. Yan, *The Euler scheme with irregular coefficients*, Ann. Probab. 30 (2002), 1172–1194.