# Besov Regularity and Approximation of a Certain Class of Random Fields

Nicolas Döhring (TU Kaiserslautern)

joint work with F. Lindner, R. Schilling (TU Dresden), K. Ritter (TU Kaiserslautern), T. Raasch (Uni Mainz), P. Cioica, S. Kinzel, S. Dahlke (Uni Marburg)

supported by DFG priority programme 1324

# The Problem

Consider a random function

$$X : \Omega \to L_2(D)$$

with a bounded Lipschitz domain $D \subset \mathbb{R}^d$.

# The Problem

Consider a random function

$$X : \Omega \to L_2(D)$$

with a bounded Lipschitz domain $D \subset \mathbb{R}^d$.

Computational task:

- approximation of $X$.

In particular

- linear vs. nonlinear approximation.

# Motivation

A centered Gaussian process $X$ admits the Karhunen-Loève decomposition

$$X(u) = \sum_{i\in\mathbb{N}} Z_i e_i(u)$$

with an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $L_2(D)$ and independent, centered Gaussian random variables $(Z_i)_{i\in\mathbb{N}}$.

# Motivation

A centered Gaussian process $X$ admits the Karhunen-Loève decomposition

$$X(u) = \sum_{i \in \mathbb{N}} Z_i e_i(u)$$

with an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of $L_2(D)$ and independent, centered Gaussian random variables $(Z_i)_{i \in \mathbb{N}}$.

Modify the representation, so that it is possible to

- introduce sparsity,
- work with wavelets.

# The Model

## Definition

A family $(e_i)_{i\in\mathbb{N}} \in L_2(D)$ ist called Riesz basis, if and only if

$$\forall f \in L_2(D)\ \exists_1 (c_i) \in \ell^2(\mathbb{N}) \qquad f = \sum_i c_i e_i$$

and $\exists A, B > 0 \quad \forall f \in L_2(D)$

$$A||c||_{\ell^2} \leq ||f||_{L_2(D)} \leq B||c||_{\ell^2}.$$

# The Model

## Definition

A family $(e_i)_{i\in\mathbb{N}} \in L_2(D)$ ist called Riesz basis, if and only if

$$\forall f \in L_2(D)\ \exists_1 (c_i) \in \ell^2(\mathbb{N}) \qquad f = \sum_i c_i e_i$$

and $\exists A, B > 0 \quad \forall f \in L_2(D)$

$$A||c||_{\ell^2} \leq ||f||_{L_2(D)} \leq B||c||_{\ell^2}.$$

From now on let $\Psi = (\psi_{j,k})$ be an $L_2(D)$ (wavelet) Riesz basis with

- dyadic refinement level $j \geq j_0$,
- spatial location $k \in \nabla_j$, with $|\nabla_j| \asymp 2^{jd}$.

# The Model

Let $Y_{j,k}, Z_{j,k}$ be a family of independent random variables with

- $P(Y_{j,k} = 1) = 1 - P(Y_{j,k} = 0) = 2^{-\beta jd}, \; 0 \leq \beta \leq 1,$
- $Z_{j,k} \sim \mathcal{N}(0, \sigma_j^2)$ with $\sigma_j^2 = 2^{-\alpha jd}, \; \alpha > 0,$
- $\alpha + \beta > 1.$

# The Model

Let $Y_{j,k}, Z_{j,k}$ be a family of independent random variables with

- $P(Y_{j,k} = 1) = 1 - P(Y_{j,k} = 0) = 2^{-\beta jd}, \ 0 \leq \beta \leq 1,$
- $Z_{j,k} \sim \mathcal{N}(0, \sigma_j^2)$ with $\sigma_j^2 = 2^{-\alpha jd}, \ \alpha > 0,$
- $\alpha + \beta > 1.$

Define

$$X(u) = \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} Y_{j,k} Z_{j,k} \psi_{j,k}(u).$$

# The Model

Let $Y_{j,k}, Z_{j,k}$ be a family of independent random variables with

- $P(Y_{j,k} = 1) = 1 - P(Y_{j,k} = 0) = 2^{-\beta jd}, \ \ 0 \leq \beta \leq 1,$
- $Z_{j,k} \sim \mathcal{N}(0, \sigma_j^2)$ with $\sigma_j^2 = 2^{-\alpha jd}, \ \ \alpha > 0,$
- $\alpha + \beta > 1.$

Define

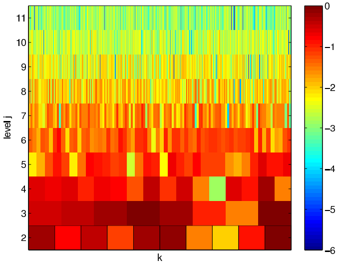
$$X(u) = \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} Y_{j,k} Z_{j,k} \psi_{j,k}(u).$$

Thus

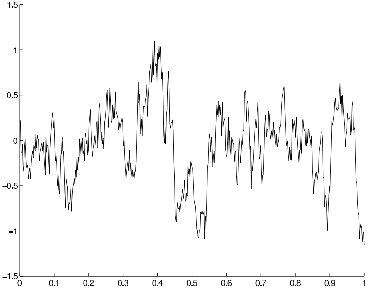
- $\alpha$ Gaussian decay parameter,
- $\beta$ sparsity parameter.

# Realisations of $X$

$\alpha = 2.0$, $\beta = 0.0$ (c.f. Brownian motion)



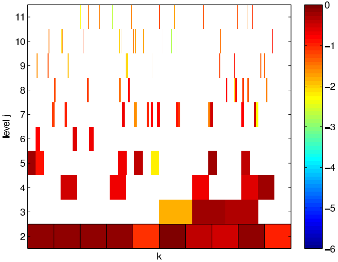

coefficients



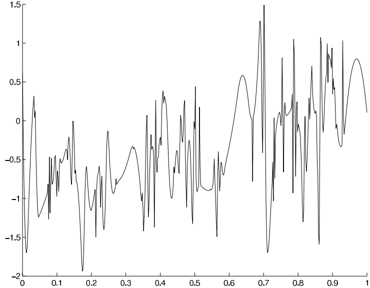

sample path

# Realisations of $X$

$\alpha = 1.2,\ \beta = 0.8$




coefficients




sample path

# Regularity of $X$

**Theorem (CDD... '10)**

*Suppose that* $s > d \cdot (\frac{1}{p} - 1)_+$. *We have*

$$X \in B^s_q(L_p(D)) \quad a.s.$$

*if and only if*

$$s < d \cdot \left( \frac{\alpha - 1}{2} + \frac{\beta}{p} \right).$$

*Furthermore* $E||X||^q_{B^s_q(L_p(D))} < \infty$.

# Regularity of $X$

**Theorem (CDD... '10)**

*Suppose that* $s > d \cdot (\frac{1}{p} - 1)_+$. *We have*

$$X \in B_q^s(L_p(D)) \quad a.s.$$

*if and only if*

$$s < d \cdot \left( \frac{\alpha - 1}{2} + \frac{\beta}{p} \right).$$

*Furthermore* $E||X||^q_{B_q^s(L_p(D))} < \infty$.

See Abramovich, Sapatinas, Silverman (1998) and Bochkina (2006) in the context of Bayesian nonparametric regression for the case $d = 1$, $p, q \geq 1$.

# Wavelet Characterisation

## Assumption

The wavelet basis induces characterisation of Besov spaces $B^s_q(L_p(D))$ of the form

$$||v||_{B^s_q(L_p(D))} \asymp \left( \sum_{j=j_0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \left( \sum_{k \in \nabla_j} |\langle v, \tilde{\psi}_{j,k} \rangle_{L_2(D)}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},$$

for $0 < p, q < \infty$ and all $s$ with $d(\frac{1}{p} - 1)_+ < s < s_1$ for some parameter $s_1 > 0$.

# Linear Approximation

**Definition**

Linear approximation error

$$e_N^{\text{lin}}(X) = \inf_{\widehat{X}} (E||X - \widehat{X}||^2_{L_2(D)})^{\frac{1}{2}},$$

where $\widehat{X} : \Omega \to L_2(D)$ measurable, such that

$$\dim(\text{span}(\hat{X}(\Omega))) \leq N.$$

# Linear Approximation

## Definition

Linear approximation error

$$e_N^{\text{lin}}(X) = \inf_{\widehat{X}} (E\|X - \widehat{X}\|_{L_2(D)}^2)^{\frac{1}{2}},$$

where $\widehat{X} : \Omega \to L_2(D)$ measurable, such that

$$\dim(\operatorname{span}(\hat{X}(\Omega))) \leq N.$$

## Theorem (CDD... '10)

$$e_N^{\text{lin}}(X) \asymp N^{-\frac{\alpha+\beta-1}{2}}.$$

# Nonlinear Approximation

For

$$f = \sum_{j,k} c_{j,k}\psi_{j,k} \in L_2(D)$$

let

$$\eta(f) = \#\left\{(j,k) : c_{j,k} \neq 0\right\}.$$

# Nonlinear Approximation

For

$$f = \sum_{j,k} c_{j,k} \psi_{j,k} \in L_2(D)$$

let

$$\eta(f) = \#\left\{(j,k) : c_{j,k} \neq 0\right\}.$$

## Definition

Best average N-term approximation error

$$e_N^{\text{non}}(X) = \inf_{\widehat{X}} (E||X - \widehat{X}||^2_{L_2(D)})^{\frac{1}{2}},$$

where $\widehat{X} : \Omega \to L_2(D)$ measurable, such that

$$E(\eta(\widehat{X})) \leq N.$$

# Nonlinear Approximation

See

- ..., DeVore (1998), ...
- Cohen, d'Ales (1997), Cohen, Daubechies, Guleryuz, Orchard (2002), Kon, Plaskota (2005), Creutzig, Müller-Gronbach, Ritter (2007), Dereich, Heidenreich (2010), ...

## Definition

Best average N-term approximation error

$$e_N^{\text{non}}(X) = \inf_{\widehat{X}} (E\|X - \widehat{X}\|^2_{L_2(D)})^{\frac{1}{2}},$$

where $\widehat{X} : \Omega \to L_2(D)$ measurable, such that

$$E(\eta(\widehat{X})) \leq N.$$

# Nonlinear Approximation

**Theorem (CDD... '10)**

$$e_N^{\text{non}}(X) \preceq \begin{cases} N^{-\frac{\alpha+\beta-1}{2(1-\beta)}}, & \textit{if } \beta \in [0,1), \\ 2^{-\frac{\alpha d N}{2}}, & \textit{if } \beta = 1. \end{cases}$$

# Nonlinear Approximation

**Theorem (CDD... '10)**

$$e_N^{\text{non}}(X) \preceq \begin{cases} N^{-\frac{\alpha+\beta-1}{2(1-\beta)}}, & \textit{if } \beta \in [0,1), \\ 2^{-\frac{\alpha d N}{2}}, & \textit{if } \beta = 1. \end{cases}$$

Recall: $e_N^{\text{lin}}(X) \asymp N^{-\frac{\alpha+\beta-1}{2}}$

# Nonlinear Approximation

**Theorem (CDD... '10)**

$$e_N^{\text{non}}(X) \preceq \begin{cases} N^{-\frac{\alpha+\beta-1}{2(1-\beta)}}, & \text{if } \beta \in [0,1), \\ 2^{-\frac{\alpha d N}{2}}, & \text{if } \beta = 1. \end{cases}$$

Recall: $$e_N^{\text{lin}}(X) \asymp N^{-\frac{\alpha+\beta-1}{2}}$$

**Remark**

- *For* $\beta \in (0,1)$ *we have* $e^{\text{non}}_{N^{1-\beta}} \preceq e^{\text{lin}}_N$.
- *Upper bound in the theorem can be achieved by an algorithm at an average computational cost of order* $E(\eta(\widehat{X}))$.

# Application

Consider the equation

$$\begin{aligned} \Delta U = X \quad &\text{in} \quad D, \\ U = 0 \quad &\text{on} \quad \partial D. \end{aligned}$$

# Application

Consider the equation

$$\Delta U = X \quad \text{in} \quad D, \qquad U = 0 \quad \text{on} \quad \partial D.$$

## Definition

Approximation error

$$e_{N,H^1(D)}(U) = \inf(E\|U - \widehat{U}\|^2_{H^1(D)})^{\frac{1}{2}},$$

where $\widehat{U} : \Omega \to H^1(D)$ measurable, such that

$$\eta(\widehat{U}) \leq N \quad \text{a.s.}$$

# Application

**Theorem (CDD... '10)**

*Let* $d \in \{2,3\}$ *and*

$$\rho = \min\left(\frac{1}{2(d-1)}, \frac{\alpha+\beta-1}{6} + \frac{2}{3d}\right).$$

*Then for every* $\varepsilon > 0$ *the error of the best* $N$*-term approximation satisfies*

$$e_{N,H^1(D)}(U) \preceq N^{-\rho+\varepsilon}.$$

# Summary

- Recall:
  - Gaussian decay parameter $\alpha$,
  - sparsity parameter $\beta$.
- Results:
  - $X \in B_q^s(L_p(D)) \Leftrightarrow s < d \cdot \left(\frac{\alpha-1}{2} + \frac{\beta}{p}\right)$.
  - Linear approximation rate determined by $\alpha + \beta$.
  - Nonlinear approximation rate determined by $\frac{1}{1-\beta}(\alpha + \beta)$.
  - For $\beta \neq 0$ nonlinear approximation pays off.

# Summary

- Recall:
  - Gaussian decay parameter $\alpha$,
  - sparsity parameter $\beta$.
- Results:
  - $X \in B^s_q(L_p(D)) \Leftrightarrow s < d \cdot \left(\frac{\alpha-1}{2} + \frac{\beta}{p}\right)$.
  - Linear approximation rate determined by $\alpha + \beta$.
  - Nonlinear approximation rate determined by $\frac{1}{1-\beta}(\alpha + \beta)$.
  - For $\beta \neq 0$ nonlinear approximation pays off.
- Application:
  - Elliptic boundary value problems with random right-hand sides.
- Work in progress:
  - lower bounds for $e^{\text{non}}_N(X)$,
  - errors in $B^s_q(L_p(D))$;
  - ...