Fast orthogonal transforms and generation of Brownian paths

G. Leobacher

Summer 2011

Discrete Brownian paths

What - and why

Many financial derivatives can be priced using

$$\text{price} = E\left(f(B)\right)$$

where B is a standard Brownian motion, $B = (B_t)_{t\in[0,1]}$ (Black-Scholes formula) or

$$\text{price} = E\left(f(B_{\frac{1}{n}}, \ldots, B_{\frac{n}{n}})\right)$$

and often the second formula is taken as an approximation for the first one.

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We therefore want efficient ways to generate a random vector

$$B = (B_{\frac{1}{n}}, \ldots, B_{\frac{n}{n}})$$

where

$$B_{\frac{1}{n}}, B_{\frac{2}{n}} - B_{\frac{1}{n}}, \ldots, B_{\frac{n}{n}} - B_{\frac{n-1}{n}}$$

are independent normal variables with mean 0 and variance $\frac{1}{n}$, from

$$X_1, \ldots, X_n \ldots i.i.d. N(0, 1)$$

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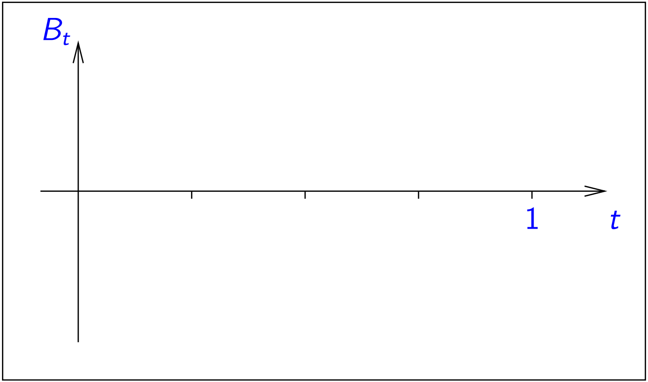
Forward construction

a.k.a. random walk construction, step-by-step method, crude method



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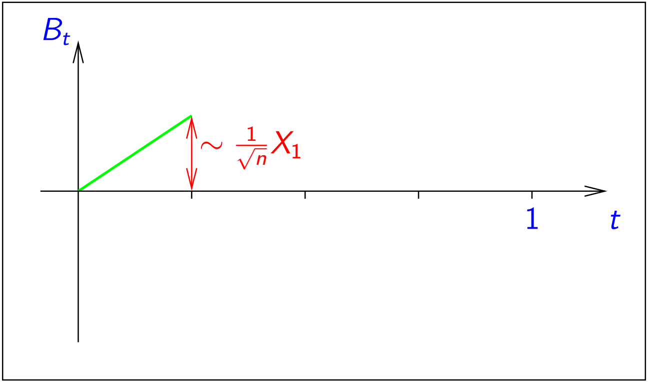

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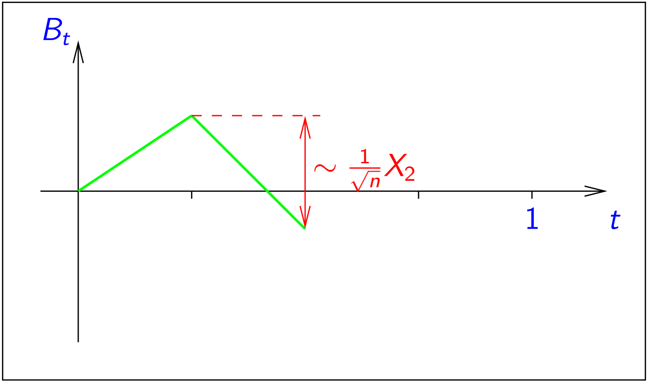

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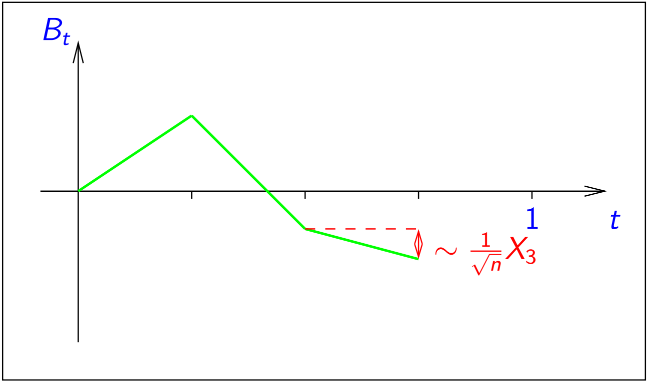

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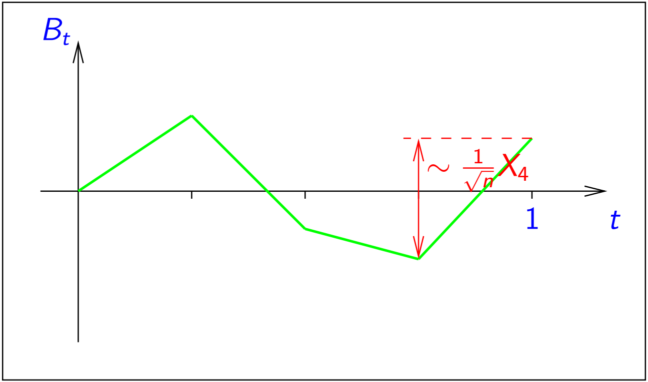

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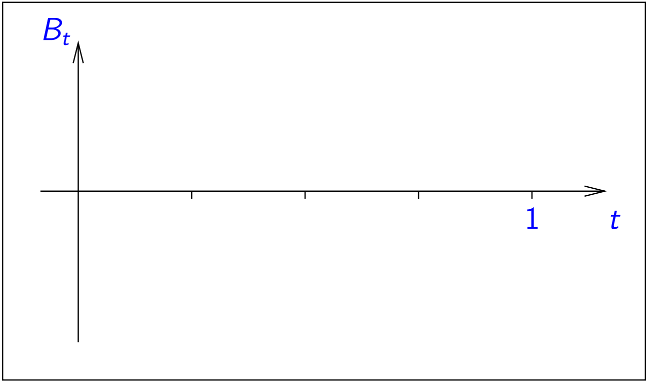
a.k.a. Lévy-Ciesielski construction, midpoint displacement



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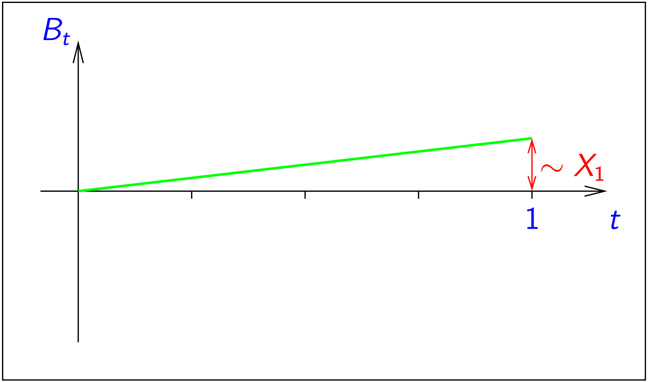

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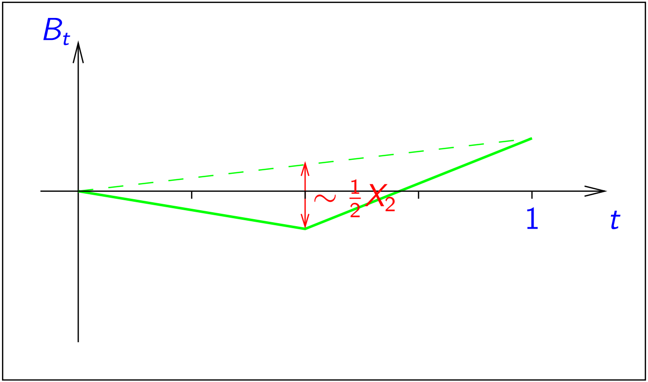

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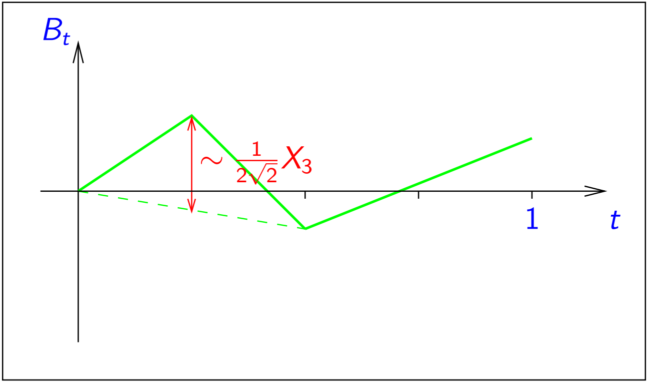

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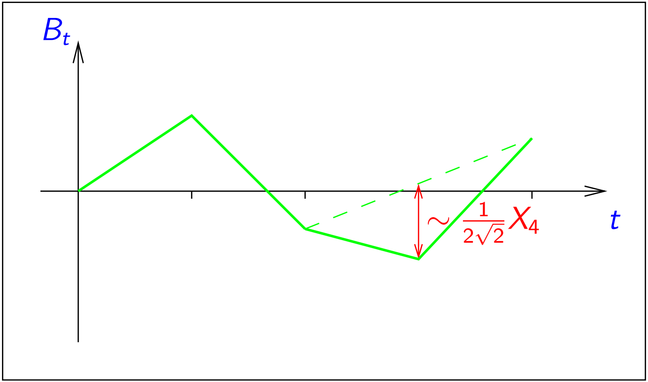

Brownian bridge construction

- Probabilistically equivalent to forward construction
- Influence of X_k on overall behavior of B is decreasing with k
  - stratified sampling
  - quasi-Monte Carlo
- Cost of generating path on n nodes is of same order as for forward construction: $O(n)$

First application in financial/QMC context: Moskowitz & Caflisch (1996)

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Unifying principle I: Linearity

Both constructions are of the form

$$B = AX$$

where

- $B = (B_{\frac{1}{n}}, \ldots, B_{\frac{n}{n}})$
- $(X_1, \ldots, X_n)$ indep. std. normal variables
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Necessary and sufficient for B being a (discrete) Brownian path:

$$AA^\top = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix} =: \Sigma$$

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- For the Brownian bridge construction

$$A = \text{something else}$$

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Principal component analysis

a.k.a. PCA construction, Singular value construction

Take

$$A = VD^{1/2}$$

where $VDV^\top = \Sigma$ is the singular value decomposition of Σ.

- First component captures maximal variance
- second component captures maximal remaining variance
- and so on

First application in financial context: Acworth, Broadie & Glasserman (1996)

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Multiplication with V costs $O(n^2)$

Or does it?

Scheicher (2007): PCA generation costs $O(n\log(n))$ by using the fast sine transform in dimension $2n+1$

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Dependence on payoff

(Some people are never satisfied)

Papageorgiou (2002), Sloan & Wang (2011)

- Integration error depends on payoff
- Let f be a payoff function, A_1, A_2 two linear generation methods such that $E(f(A_1X))$ admits good/bad convergence under some QMC rule. Then there is a payoff g such that $E(g(A_2X))$ admits the same convergence under the same rule: $g(Y) = f(A_1A_2^{-1}Y)$.
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In fact we can write any matrix A with $AA^\top = \Sigma$ as

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where

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- PCA/BB provide good U for Asian options (and many other types)

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Sample application: BB/PCA-type generation for Lévy paths (L (2006))

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- Hartley-, Hilbert-, W- transform
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Empirical finding: for typical real world derivatives in the 1-dimensional BS-model, BB/PCA are good enough

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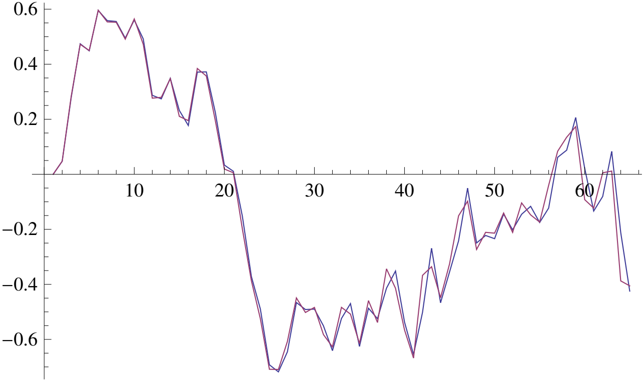

Comparison PCA/DCT-IV, $n = 64$

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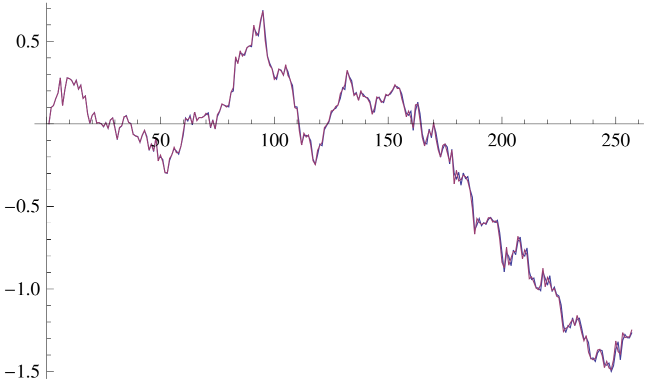

Comparison PCA/DCT-IV, $n = 256$

Unifying principle II: Orthogonality
Numerical example

Derived from Sloan & Wang 2011: weighted Asian option.

$$
\begin{aligned}
f(B) &:= \max\left(\sum_{k=1}^{n} w_k S_{k/n} - K, 0\right) \\
&= \max\left(\frac{1}{n}\sum_{k=1}^{n} w_k S_0 \exp\left(\sigma B_{k/n} + (r - \frac{\sigma^2}{2})k/n\right) - K, 0\right),
\end{aligned}
$$

Unifying principle II: Orthogonality
Numerical example

OTP ... "orthogonal tensor product"

$$U = R \otimes R \otimes \ldots \otimes R \quad (\log_2(n) \text{ times})$$

R is a two-dimensional rotation about a fixed angle ($\phi = 0.4$).

Now choose weights $w_1, \ldots, w_n$ to make U a (near) optimal orthogonal transform.

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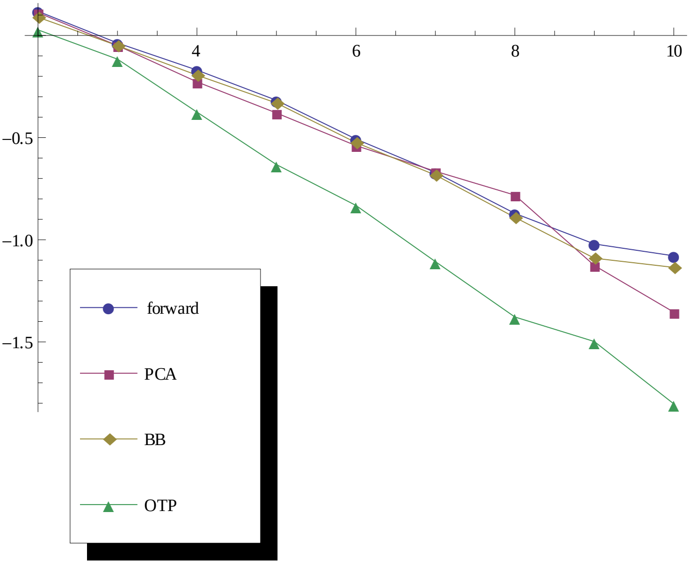

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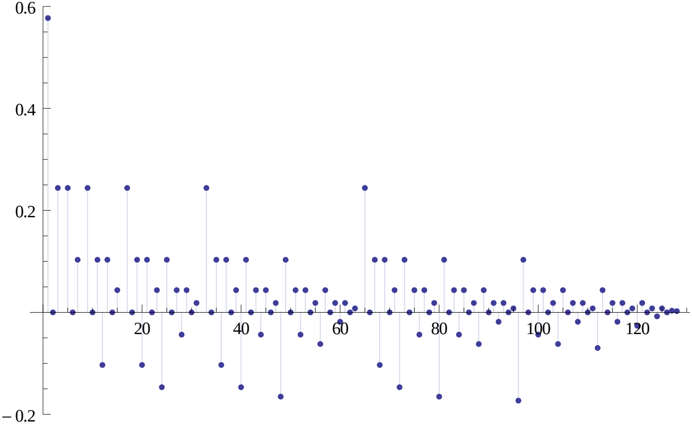
The corresponding weights are rather exotic:



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At least on of these earlier constructions is actually useful in practice

Recall: Fast computation of path using PCA utilizes fast sine transform in dimension $2n + 1$

This is usually quite a bit slower than, e.g., the discrete cosine transform in dimension n, especially if $n = 2^k$

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Theorem (L 2011)

Let

- $\Sigma = VDV^\top$ *be the PCA of* Σ
- C *the discrete cosine transform of type IV in dimension* n
- $d_n(P,Q)^2 := \sum_{l=1}^n \sum_{k=1}^n (P-Q)_{lk}^2$ *for* $n \times n$ *matrices* P, Q

Then for all $n \in \mathbb{N}$ *we have* $d_n(SC, VD^{\frac{1}{2}}) < 1$ *and*

$$\limsup_{n\to\infty} d_n(SC, VD^{\frac{1}{2}})^2 \le \frac{2\left(48-\pi^2\right)}{\left(\pi^2-24\right)^2} = 0.381\ldots .$$

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Goal: Find good U that admits fast matrix-vector multiplication
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Thank you for your attention!