Optimal Pointwise Approximation of Stochastic Heat Equations with Additive Noise

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Partially supported by the DFG

Contents

Strong Approximation of SPDEs

- *Grecksch, Kloeden (1996)*, *Gyöngy, Nualart (1997)* and **. . .** :
  Upper bounds for error of algorithms based on uniform discretization in space and time.

- *Davie, Gaines (2001)* and *Müller-Gronbach, Ritter (2007)*:
  Lower bounds for classes of algorithms, optimality.
  Non-uniform time discretization.

Stochastic Heat Equation with Additive Noise

$$dX(t) = \Delta X(t)\, dt + B(t)\, dW(t), \quad t \in (0, T],$$
$$X(0) = 0,$$

with Dirichlet boundary conditions and

- $W(t) = \sum_{i \in \mathbb{N}^d} |i|_2^{-\frac{\gamma}{2}} \cdot \beta_i(t) \cdot h_i$
  Wiener process in $H = L_2((0,1)^d)$ in which
  - (β_i) independent standard one-dimensional Brownian motions,
  - $h_i(u) = 2^{\frac{d}{2}} \prod_{k=1}^{d} \sin(i_k \pi u_k)$ eigenfunction of Δ for $i \in \mathbb{N}^d$,
- B operator-valued mapping.

Case 1 (TC(γ)) $\gamma > d$.
Case 2 (ID) $\gamma = 0$ and $d = 1$.

Computational Problem

Task

Approximate $X(T)$ based on evaluations of finitely many scalar Brownian motions β_i's at a finite number of points.

Error and cost of any approximation $\widehat{X}(T)$

$$e\left(\widehat{X}(T)\right) = \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|_H^2\right)^{1/2},$$

$$\mathrm{cost}\left(\widehat{X}(T)\right) = \text{total number of evaluations of the } \beta_i\text{'s.}$$

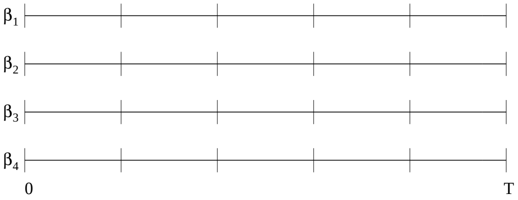
Goal

Approximation with optimal relation between error and cost.

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^{\text{uni}}$ of algorithms with **uniform** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- use nodes $t_k = \frac{k}{n} T$, $k = 1, \ldots, n$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.



Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^{\text{uni}}$ of algorithms with **uniform** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- use nodes $t_k = \frac{k}{n} T$, $k = 1, \dots, n$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.

Approximation:

$$\widehat{X}_N^{\text{uni}}(T) = \phi(\beta_{i_1}(t_1), \dots, \beta_{i_1}(t_n), \dots, \beta_{i_\ell}(t_1), \dots, \beta_{i_\ell}(t_n))$$

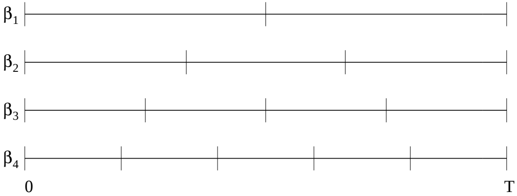
for $\mathcal{I} = \{i_1, \dots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}_N^{\text{uni}}(T)\right) = N.$$

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^{\text{equi}}$ of algorithms with **equidistant** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n_i \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- use nodes $t_{k,i} = \frac{k}{n_i} T$, $k = 1, \ldots, n_i$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.



Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^{\text{equi}}$ of algorithms with **equidistant** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n_i \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- use nodes $t_{k,i} = \frac{k}{n_i} T$, $k = 1, \dots, n_i$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Approximation:

$$\widehat{X}_N^{\text{equi}}(T) = \phi(\beta_{i_1}(t_{1,i_1}), \dots, \beta_{i_1}(t_{n_{i_1},i_1}), \dots, \beta_{i_\ell}(t_{1,i_\ell}), \dots, \beta_{i_\ell}(t_{n_{i_\ell},i_\ell}))$$

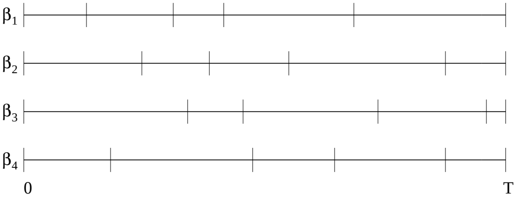
for $\mathcal{I} = \{i_1, \dots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}_N^{\text{equi}}(T)\right) = N.$$

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^{\text{noeq}}$ of algorithms with **non-equidistant** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $0 < t_{1,i} < \ldots < t_{n,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.



Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^{\text{noeq}}$ of algorithms with **non-equidistant** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $0 < t_{1,i} < \ldots < t_{n,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.

Approximation:

$$\widehat{X}_N^{\text{noeq}}(T) = \phi(\beta_{i_1}(t_{1,i_1}), \ldots, \beta_{i_1}(t_{n,i_1}), \ldots, \beta_{i_\ell}(t_{1,i_\ell}), \ldots, \beta_{i_\ell}(t_{n,i_\ell}))$$

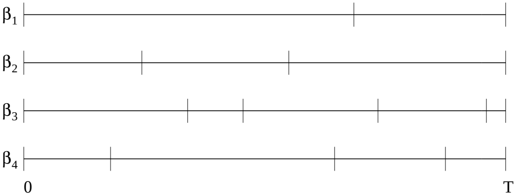
for $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}_N^{\text{noeq}}(T)\right) = N.$$

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^*$ of algorithms with **arbitrary** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n_i \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.



Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_N^*$ of algorithms with **arbitrary** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n_i \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Approximation:

$$\widehat{X}_N^*(T) = \phi(\beta_{i_1}(t_{1,i_1}), \ldots, \beta_{i_1}(t_{n_{i_1},i_1}), \ldots, \beta_{i_\ell}(t_{1,i_\ell}), \ldots, \beta_{i_\ell}(t_{n_{i_\ell},i_\ell}))$$

for $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}_N^*(T)\right) = N.$$

Classes of Algorithms and Minimal Errors

***N*-th minimal error** with $\diamond \in \{*, \text{noeq}, \text{equi}, \text{uni}\}$

$$\mathrm{e}_N^{\diamond} = \inf \left\{ \mathrm{e}\left(\widehat{X}_N^{\diamond}(T)\right) \;\middle|\; \widehat{X}_N^{\diamond}(T) \in \mathfrak{X}_N^{\diamond} \right\}.$$

Clearly

$$\mathrm{e}_N^{*} \leq \mathrm{e}_N^{\text{noeq}} \leq \mathrm{e}_N^{\text{uni}}$$

and

$$\mathrm{e}_N^{*} \leq \mathrm{e}_N^{\text{equi}} \leq \mathrm{e}_N^{\text{uni}}.$$

Questions:

- Rate of convergence of $\mathrm{e}_N^{\diamond}$?
  Superiority of $\mathfrak{X}_N^{*}$ over $\mathfrak{X}_N^{\text{noeq}}$, $\mathfrak{X}_N^{\text{equi}}$ over $\mathfrak{X}_N^{\text{uni}}$?
- Construction of (asymptotically) optimal algorithms $\widehat{X}_N^{\diamond}(T) \in \mathfrak{X}_N^{\diamond}$?

Classes of Algorithms and Minimal Errors

Notation:
Let $x, y \in (0, \infty)^{\mathbb{N}}$.

$$x_N \preceq y_N \quad \text{if } \sup_{N \in \mathbb{N}} \frac{x_N}{y_N} < \infty.$$

$$x_N \asymp y_N \quad \text{if } x_N \preceq y_N \text{ and } y_N \preceq x_N.$$

Results for Equations with Additive Noise

Theorem *Henkel, Müller-Gronbach, Ritter, Wagner*
Assume (ID) and $B = id$. Then

$$\begin{aligned} \mathrm{e}_N^{\mathrm{equi}} \asymp \mathrm{e}_N^{\mathrm{uni}} &\asymp N^{-1/6}, \\ \mathrm{e}_N^{\mathrm{noeq}} &\asymp N^{-1/3}, \\ \mathrm{e}_N^{*} &\asymp N^{-1/2}. \end{aligned}$$

Results for Equations with Additive Noise

Theorem *Henkel, Müller-Gronbach, Ritter, Wagner*
Assume (TC(γ)) and $B = id$. Then

$$e_N^{\mathrm{uni}} \asymp \begin{cases} N^{-(\gamma-d+2)/(2d+4)}, & \text{if } \gamma < d+2, \\ N^{-(\gamma-d+2)/(\gamma+d+2)}, & \text{if } \gamma > d+2, \end{cases}$$

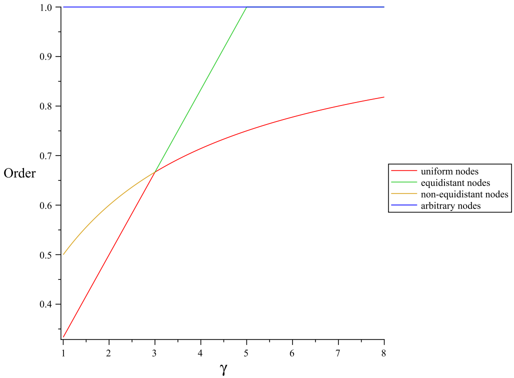
$$e_N^{\mathrm{equi}} \asymp \begin{cases} N^{-(\gamma-d+2)/(2d+4)}, & \text{if } \gamma < 3d+2, \\ N^{-1}, & \text{if } \gamma > 3d+2, \end{cases}$$

$$e_N^{\mathrm{noeq}} \asymp N^{-(\gamma-d+2)/(\gamma+d+2)},$$

$$e_N^{*} \asymp \begin{cases} N^{-(\gamma-d+2)/(2d)}, & \text{if } \gamma < 3d-2, \\ N^{-1}, & \text{if } \gamma > 3d-2. \end{cases}$$

Results for Equations with Additive Noise

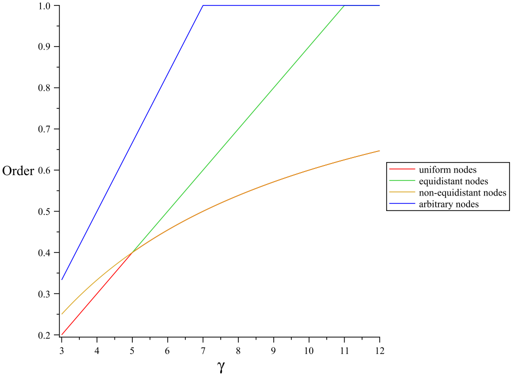
Theorem *Henkel, Müller-Gronbach, Ritter, Wagner*
Assume $(\mathrm{TC}(\gamma))$ and $B = id$. Then



$$(\mathrm{TC}(\gamma)), \ d = 1$$

Results for Equations with Additive Noise

Theorem *Henkel, Müller-Gronbach, Ritter, Wagner*
Assume (TC(γ)) and $B = id$. Then



$$(\mathrm{TC}(\gamma)),\ d = 3$$

Results for Equations with Additive Noise

Assumption (A(α)): Put $B_{i,j}(t) = \langle B(t)h_i, h_j \rangle$.
For $d = 1$ and $\alpha > 1$ assume

$$\sup_{t\in[0,T]} \left((B_{i,j}(t))^2 + (B'_{i,j}(t))^2\right) \preceq \frac{1}{|i-j|^\alpha + 1} \qquad (1)$$

$$\text{and} \qquad \inf_{t\in[0,T]} (B_{i,i}(t))^2 > 0. \qquad (2)$$

Example: If $\alpha = 2$, (1) holds for multiplication operators $B(t)$, i.e.

$$(B(t)h)(u) = G(t,u) \cdot h(u)$$

with $G \in C^{(1,1)}([0,T] \times [0,1])$.

Results for Equations with Additive Noise

Theorem *Henkel*
Assume (ID) and (A(α)). Then

$$N^{-1/6} \preceq \quad e_N^{\mathrm{uni}} \quad \preceq \begin{cases} N^{-(\alpha-1)/(4\alpha-2)}, & \text{if } 1 < \alpha < 2, \\ N^{-1/6}, & \text{if } 2 \leq \alpha < \infty, \end{cases}$$

$$N^{-1/3} \preceq \quad e_N^{\mathrm{noeq}} \quad \preceq \begin{cases} N^{-(\alpha-1)/(2\alpha)}, & \text{if } 1 < \alpha < 2, \\ N^{-1/4}, & \text{if } 2 \leq \alpha < \infty. \end{cases}$$

- Suboptimality of $\mathfrak{X}_{\mathrm{uni}}$ (at least), if $\alpha > \frac{3}{2}$.

Results for Equations with Additive Noise

Theorem *Henkel*
Assume (TC(γ)) and (A(α)). Then for $\alpha = 2$

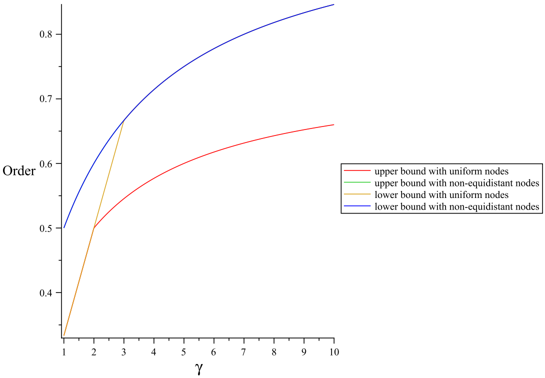
$$\left.\begin{array}{ll} N^{-(\gamma+1)/6}, & \text{if } \gamma < 3, \\ N^{-(\gamma+1)/(\gamma+3)}, & \text{if } \gamma > 3, \end{array}\right\} \preceq \mathrm{e}_N^{\mathrm{uni}} \preceq \begin{cases} N^{-(\gamma+1)/6}, & \text{if } \gamma < 2, \\ N^{-(3\gamma+3)/(4\gamma+10)}, & \text{if } \gamma > 2, \end{cases}$$

$$\mathrm{e}_N^{\mathrm{noeq}} \asymp N^{-(\gamma+1)/(\gamma+3)}.$$

- Suboptimality of $\mathfrak{X}_{\mathrm{uni}}$, if $\gamma < 3$.

Results for Equations with Additive Noise

Theorem *Henkel*
Assume (TC(γ)) and (A(α)). Then for $\alpha = 2$



$$(\mathsf{TC}(\gamma)),\ d = 1,\ \alpha = 2$$

Results for Equations with Additive Noise

Remark Upper bounds for $e_N^\diamond$

- Time discretizations are quantiles of the density $t \mapsto \exp(-\frac{\mu_j}{3}(T-t))$, i.e.

$$\int_0^{s_{k,j}} \exp\left(-\frac{\mu_j}{3}(T-t)\right)\,dt = \frac{k}{\nu_j}\int_0^T \exp\left(-\frac{\mu_j}{3}(T-t)\right)\,dt$$

  for $j \in \mathcal{J} \subset \mathbb{N}^d$, $\mu_j = \pi^2 |j|_2^2$, $\nu_j \in \mathbb{N}$, $k = 1, \ldots, \nu_j$ and $\{t_{1,i}, \ldots, t_{n,i}\} = \bigcup_{j \in \mathcal{J}} \{s_{1,j}, \ldots, s_{\nu_j,j}\}$ for every β_i.
- Drift-implicit Euler-Maruyama scheme.

Summary and Outlook

Summary

For additive noise with decay condition (A(α)) the minimal error e_N^{noeq} is superior to e_N^{uni}, if

(ID) $\alpha > \frac{3}{2}$.

(TC(γ)) $d = 1$, $\alpha = 2$, $\gamma < 3$.

Outlook

- $d = 1$, $\alpha \neq 2$.
- $d \geq 2$.
- Sharp bounds for every $e_N^{\diamond}$