Monte Carlo Simulation of Vehicular Traffic Flow for Kinetic Model with Distance Oriented Interactions

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Introduction

Kinetic Models

Within the kinetic theory VTF is considered as a gas, i.e. as a system of interacting particles. Every particle in this system corresponds to a certain vehicle. These models are called mesoscopic due to the fact that they use information of a single vehicle behaviour as input and produce results of the whole traffic flow.

The most obvious differences of VTF from gas flow are:

- it is organized and mostly one dimensional;
- it is based on a deterministic set of rules;
- it depends (more or less) on individual drivers.

Main quantity of interest: one-particle probability density $f(x, v, a, t)$, which describes the state of a car.

$f(x, v, a, t)dxdvda$ is the probability to find a car at place between x and $x + dx$, with velocity between v and $v + dv$, with acceleration between a and $a + da$ at time t.

Boltzmann kinetic equation:

$$\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\nabla_r + \mathbf{F} \cdot \nabla_p\right] f(\mathbf{r}, \mathbf{p}; t) = \left(\frac{\partial f}{\partial t}\right)_{coll}.$$

The item $\left(\frac{\partial f}{\partial t}\right)_{coll}$ is responsible for particle interactions in the system.

Describing VTF we assume, that the state of a vehicle is determined by a number of characteristics (x, v, a, ...).

We say that an interaction took place between two vehicles, if there was any change in the state of these vehicles.

Assumptions for interactions

- only pair interactions of cars are considered;
- characteristics of leading car are not changed due to an interaction;
- duration of an interaction $\ll$ time between two interactions;
- the pair state density for two cars decouples into a product of two single car densities (*vehicular chaos*).

Acceleration Oriented Kinetic Model (K.T.Waldeer)

The time scale of changes in acceleration is much smaller than time scale of any other relevant process. In the model, suggested by K.T. Waldeer, the acceleration variable is regarded as one of the phase coordinates, describing the state of a car. So, we assume that

- acceleration of the following car is changed discontinuous.

For a **spatial homogeneous** traffic flow the **Boltzmann-like equation** describing VTF has the following form

$$\frac{\partial f}{\partial t}+a\frac{\partial f}{\partial v}=\int\limits_{\bar{v},\bar{a},a'}\left[\Sigma(a|v,a',\bar{v},\bar{a})f(a',v,t)-\Sigma(a'|v,a,\bar{v},\bar{a})f(a,v,t)\right]f(\bar{a},\bar{v},t)\,\mathtt{d}\bar{a}\,\mathtt{d}\bar{v}\,\mathtt{d}a'$$

- We distinguish two types of vehicles: the **leader** ($\bar{v}$, $\bar{a}$) and the **follower**.
- The function $\Sigma(a' \to a|v,\bar{a},\bar{v})$ is **weighted interaction rate function** (*which determines the type of interaction in the system*), we introduce it later.
- **Boundary condition:**
  - there are no cars with negative velocities;
  - there is a maximum velocity of a VTF, which can not be exceeded.

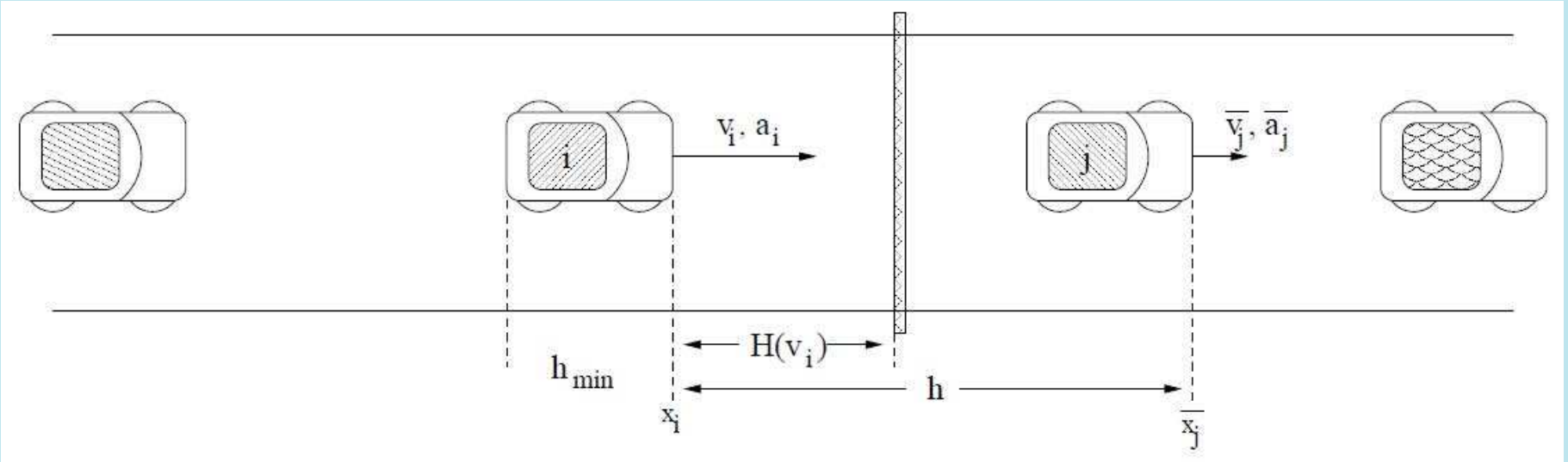
Interaction rate function $\Sigma(a' \to a|v, \bar{a}, \bar{v})$:

$$\Sigma(a|a', v, \bar{a}, \bar{v}, \mathbf{m}_f(t)) = \int_{h_{\min}}^{\infty} \sigma(a|h, a', v, \bar{a}, \bar{v}) \cdot Q(h, a', v, \bar{a}, \bar{v}) \cdot D(h|a', v, \mathbf{m}_f(t)) \, \mathrm{d}h.$$

- $h_{\min}$ is the minimal distance between two cars at rest, i. e. the **mean length of a car**;
- $Q(\cdot)$ is the **interaction rate**, it depends on a current microscopic state of the interacting car pair and the distance h between them;
- $\sigma(\cdot)$ is the **probability density of the follower's acceleration** in case when the interaction between the cars with states (a', v) and $(\bar{a}, \bar{v})$ takes place at distance h;
- $D(\cdot)$ is a **conditioned probability density of the distance** h. It depends on the follower's state (a', v) and a vector $\mathbf{m}_f(t)$, which value is determined by some moments of the solution f (such as mean velocity, velocity scattering, mean acceleration, etc.). Further on the function $D(\cdot)$ will also depend on the car density $\mathcal{K}$.

Distance Oriented Interactions

We consider an interaction model with dependence on the distance between cars, and study stochastic equilibrium distribution with respect to car density $\mathcal{K}$.



- Let us consider simple **threshold function** $H(v) = \alpha \cdot v + h_{\min}$.
- **Density** $\sigma(\cdot)$ with two possible accelerations a:

$$\sigma(a|h, v) = \Theta(h - H(v)) \cdot \delta(a - a_0) + \Theta(H(v) - h) \cdot \delta(a + a_0).$$

- We choose constant **interaction rate**: $Q = 1/\mathcal{T}$.

- Exponential distribution is a good approximation for the density $D(\cdot)$:

$$D(h) = \frac{1}{\bar{H} - h_{\min}} \exp\left\{-\frac{h - h_{\min}}{\bar{H} - h_{\min}}\right\} \Theta(h - h_{\min}),$$

here $\bar{H}$ is mean distance between interacting cars, i.e. $\bar{H} = 1/\mathcal{K}$.

- Taking these functions into account we find the form of Σ:

$$\Sigma(a_i' \to a_i | v_i, a_j, v_j) = p \cdot Q \cdot \delta(a + a_0) + (1 - p) \cdot Q \cdot \delta(a - a_0),$$

where $p = \mathbb{P}(h < H(v_i)) = \int\limits_{h_{\min}}^{H(v_i)} D(h)\, \mathtt{d}h.$

- We can solve equation analytically for given $\Sigma(\cdot)$:

$$f(v) = \frac{\exp\left\{-\frac{v}{a_0 \mathcal{T}} - 2\beta \mathrm{e}^{-\frac{v}{a_0 \mathcal{T} \beta}}\right\}}{a_0 \mathcal{T} \left(\mathrm{e}^{-2\beta} + \beta (2\beta)^{-\beta} \gamma(\beta, 2\beta)\right)}, \quad \beta = \frac{\bar{H} - h_{\min}}{\alpha a_0 \mathcal{T}}.$$

Integral Equation of the Second Kind

For the given ensemble of N cars we can obtain the following integral equation of the second kind in the phase space with coordinates $(Z,t) = (\pi = (i,j), A, V, t)$ for $F(Z,t)$:

$$F(Z,t) = \delta(t)P_0(A,V)\delta(\pi_0) + \int_0^t \int F(Z',t')K(Z',t' \to Z,t)\,\mathtt{d}Z'\,\mathtt{d}t'.$$

(or $F = \mathbf{K}F + F_0$) with the following multiplicative kernel K:

$$K(Z',t' \to Z,t) = K_t(t' \to t|A',V')K_V(V' \to V|A',t-t')K_\pi(\pi)K_a(a_i' \to a_i|\pi,V).$$

$F(Z,t)$ is a **distribution density of the interactions in the system**.

The transition in **Markov chain**, which is related to the integral equation, consists of several elementary transitions in the following order:

$$(A', V', t') \to (A', V', t) \to (A', V, t) \xrightarrow{\pi} (A, V, t)$$

The transition of the system from the state Z' to the state Z is performed as follows:

1. the instant t of the next interaction in the system is chosen according to the exponential transition density

$$K_t(t' \to t | A', V') = \chi(t' < t)\nu(A', V' + A'(t - t'))\mathrm{e}^{\left\{-\int\limits_{t'}^{t} \nu(A', V' + A'(\tau - t'))\,\mathrm{d}\tau\right\}},$$

$$\nu(A, V) = \frac{1}{N-1}\sum_{i \neq j}\int \Sigma(a_i \to a_i'' | v_i, v_j, a_j)\,\mathrm{d}a_i'' = \sum_{\pi}\frac{\nu_{(i,j)}}{N-1};$$

2. the velocities of all cars are calculated at time t according to the transition density

$$K_V(V' \to V|A', t - t') = \delta(V - V' - A'(t - t'));$$

3. the pair number (i, j) is chosen by the probabilities

$$K_\pi(i, j) = \frac{1}{N - 1} \cdot \frac{\nu_{(i,j)}}{\nu(A', V)};$$

4. new accelerations of all cars are determined as follows:

   - for the car with number i (the follower in the pair (i, j)) its acceleration a_i is changed according to the transition density

$$K_a(a_i' \to a_i|\pi, V) = \Sigma(a_i' \to a_i|v_i, a_j, v_j)/\nu_{(i,j)};$$

   - the accelerations of other cars do not change.

Estimation of Functionals

- The following functionals of one-particle distribution function f are of our interest:

$$I_{\mathbf{h}}(T) = \int\int \mathbf{h}(v,a) f(T,v,a)\, \mathtt{d}v\, \mathtt{d}a = (\mathbf{h}, f).$$

- We can prove that

$$I_{\mathbf{h}}(T) = \int_0^T \mathbf{h}_N(A, V + A(T - t'))\, \exp\left\{ -\int_{t'}^{T} \nu(A, V + A(\tau - t')\, \mathtt{d}\tau \right\} F(Z, t') \mathtt{d}Z \mathtt{d}t',$$

where $\mathbf{h}_N(V,A) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{h}(v_i, a_i)$. As a result we have

$$I_{\mathbf{h}}(T) = (\tilde{\mathbf{h}}_N, F).$$

- For numerical estimation of $I_{\mathbf{h}}(T)$ we can use the collision estimator or absorption estimator, which are functionals of the Markov chain trajectory.

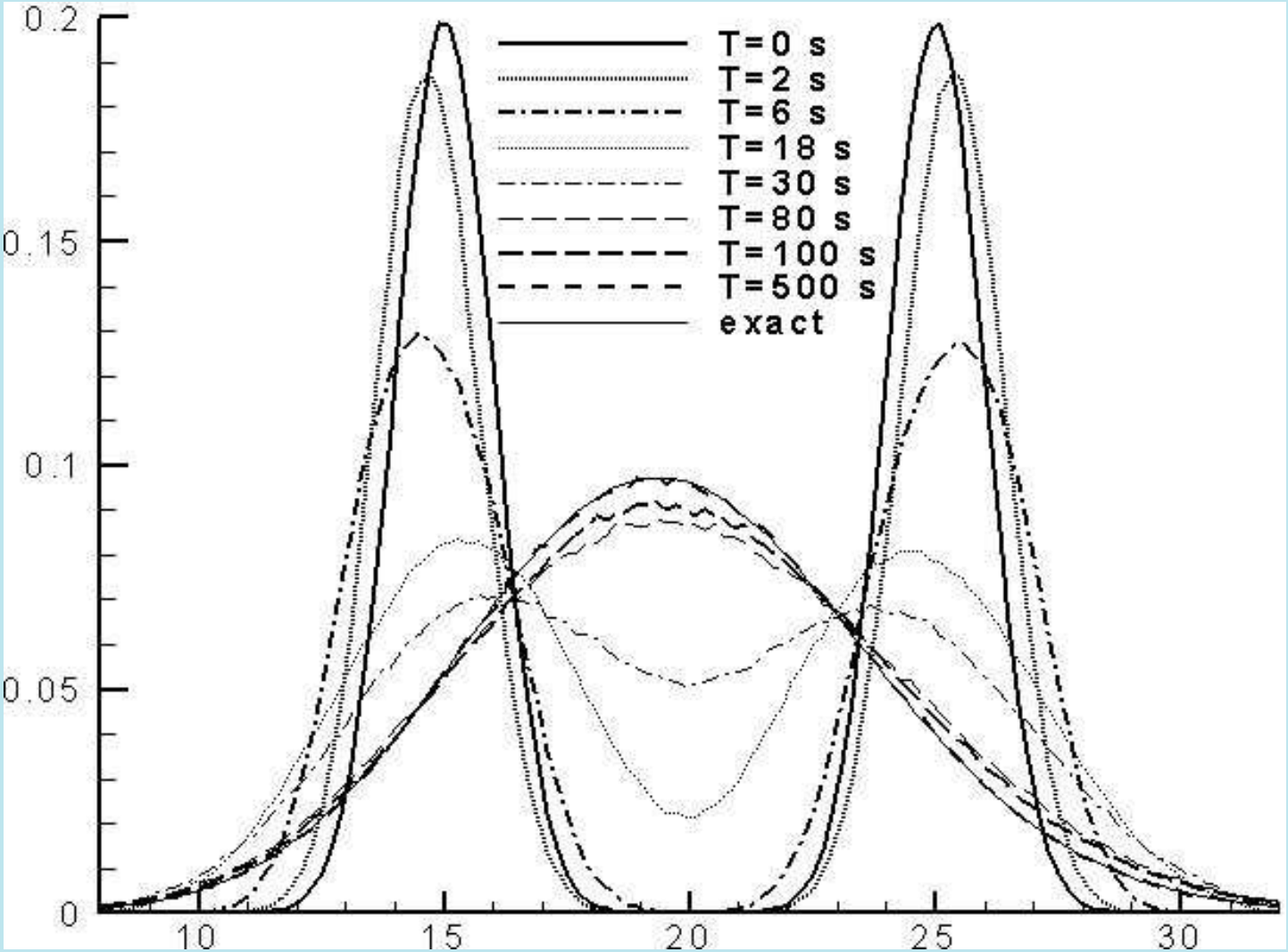


Figure 1: Numerical estimate of the velocity distribution evolution $f(v)$. $\mathcal{T} = 2.5$ s, $h_{\min} = 6.5$ m, $\mathcal{K} = 0.025$ m^{-1}, $a_0 = 0.3$ m/s^2, $\alpha = 1.2$ s, $\sigma_0 = 1$ m/s.

- Next we will estimate
  - the mean velocity;
  - the mean square scattering of the velocity;
  - the flow density.
- We simulated $M = 10^2$ trajectories of the system of $N = 10^2$ vehicles, using the following parameters:

$$\mathcal{T} = 2.5 \text{ s}, \;\; h_{\min} = 6.5 \text{ m},$$
$$\alpha_1 = 1.2 \text{ s}, \;\; \alpha_2 = 1.5 \text{ s}, \;\; \alpha_3 = 1.8 \text{ s},$$
$$a_{01} = 0.05 \text{ m/s}^2, \;\; a_{02} = 0.1 \text{ m/s}^2, \;\; a_{03} = 0.2 \text{ m/s}^2.$$

- In Fig. 2-4 we present the estimates of functionals for nine sets of the parameters (α, a_0). The correspondence of the curves in the graphs is the following:

$$1 \sim (\alpha_1, a_{01}), \;\; 2 \sim (\alpha_1, a_{02}), \;\; 3 \sim (\alpha_1, a_{03}),$$
$$4 \sim (\alpha_2, a_{01}), \;\; 5 \sim (\alpha_2, a_{02}), \;\; 6 \sim (\alpha_2, a_{03}),$$
$$7 \sim (\alpha_3, a_{01}), \;\; 8 \sim (\alpha_3, a_{02}), \;\; 9 \sim (\alpha_3, a_{03}).$$

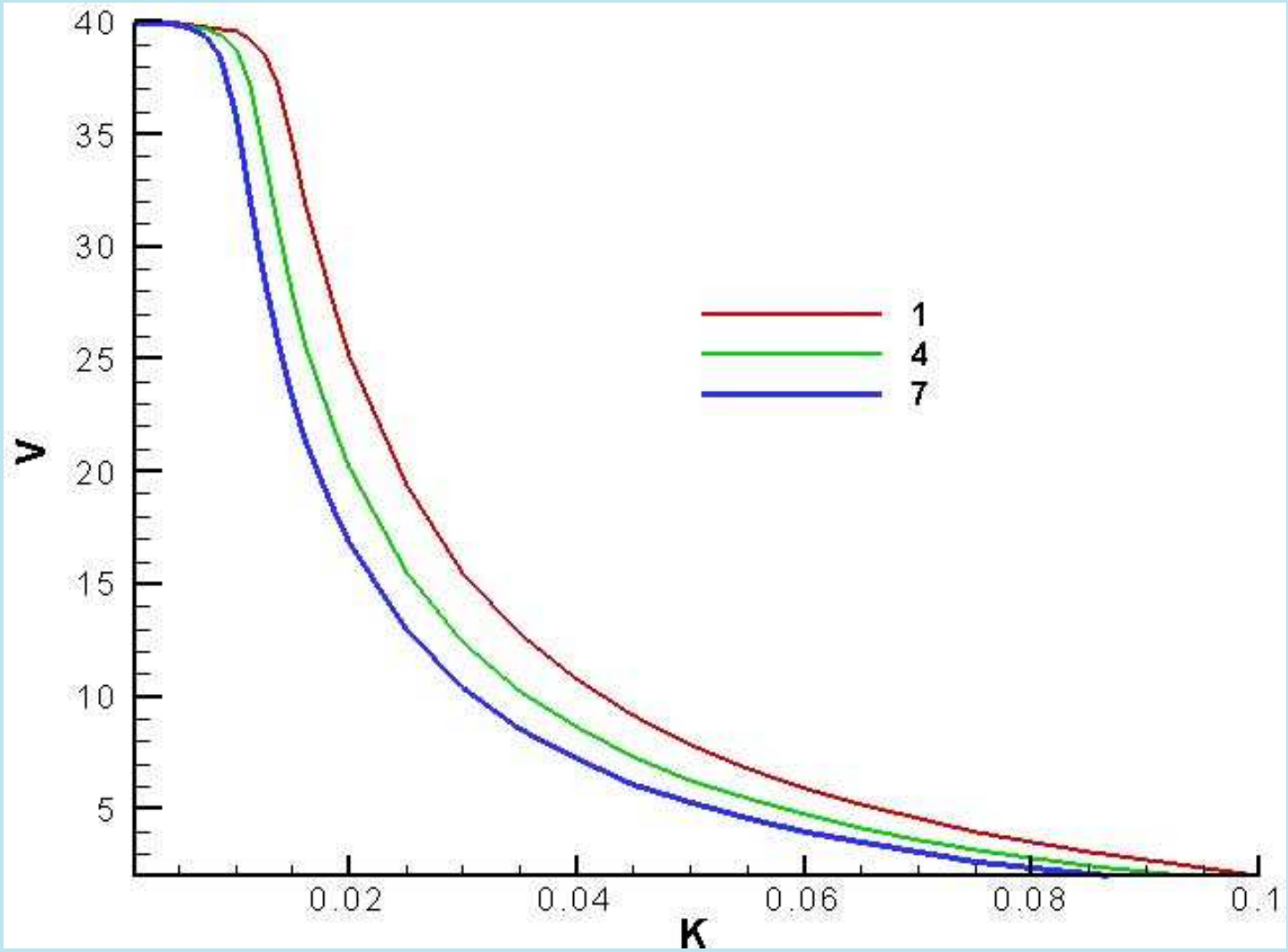


Figure 2: Numerical estimate of the mean velocity V dependence on car density $\mathcal{K}$. ($\mathbf{h}(a, v) = v$)

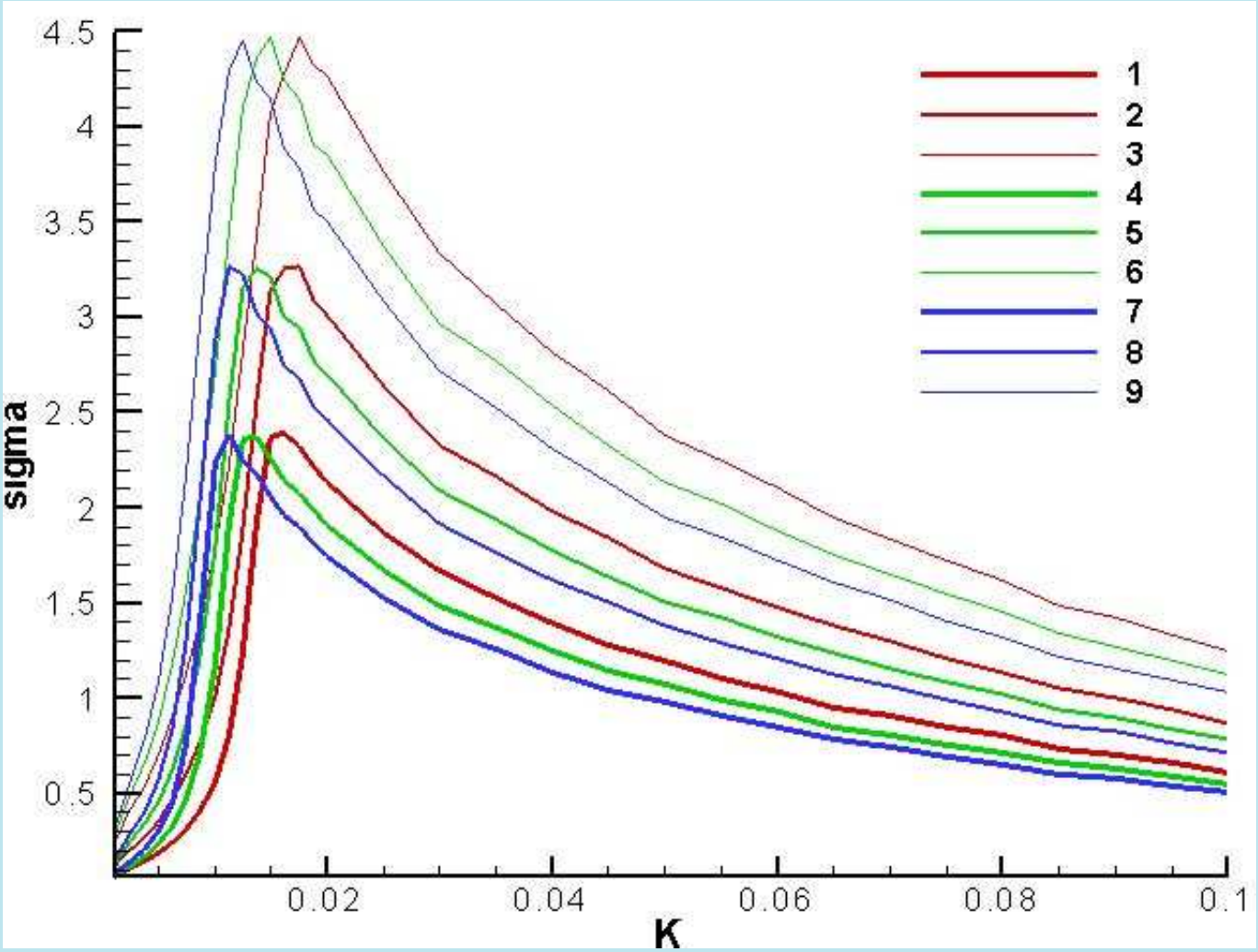


Figure 3: Numerical estimate of the mean square scattering σ_{V} dependence on car density $\mathcal{K}$. $(\mathbf{h}(a, v) = v^2)$

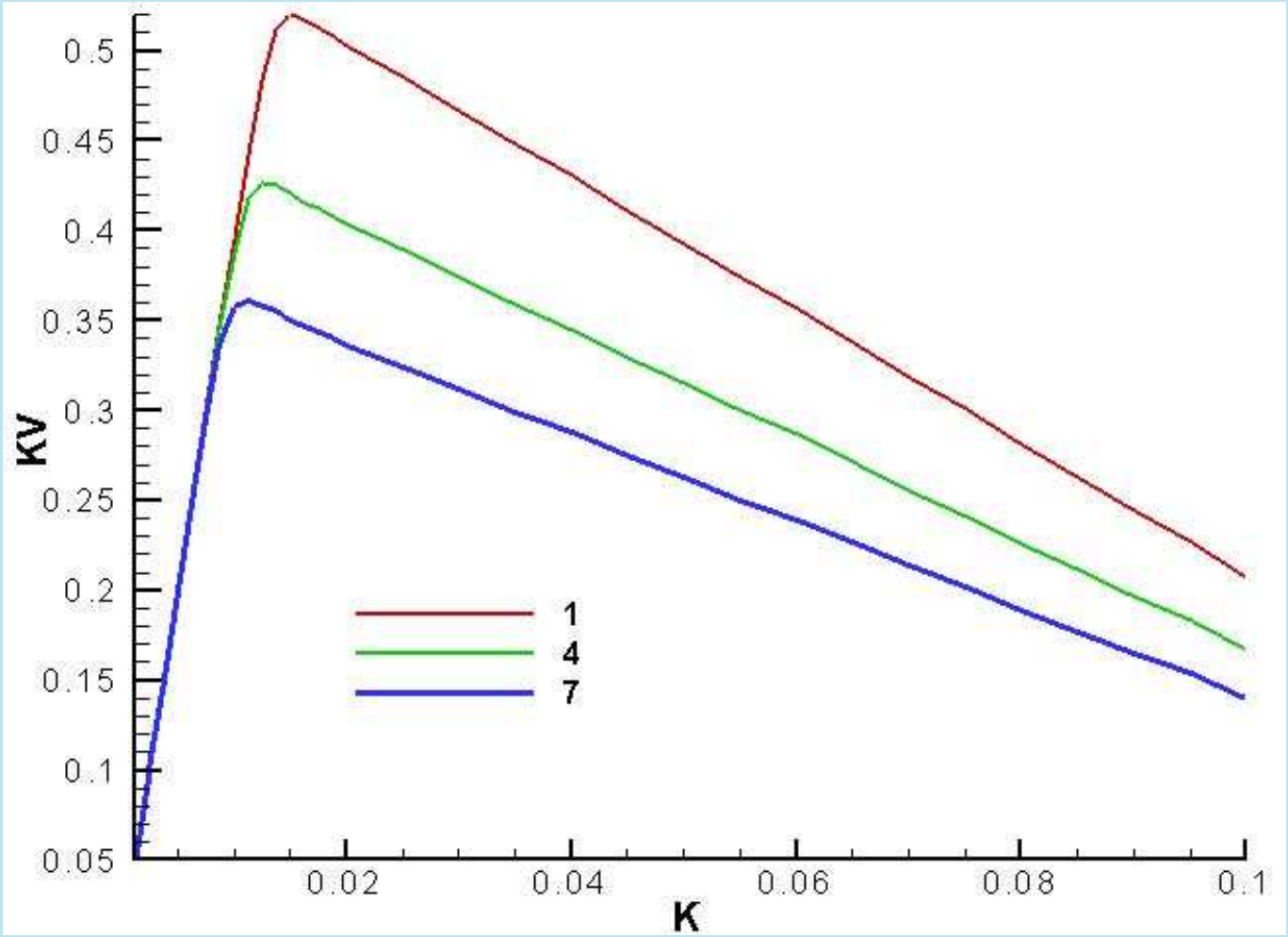


Figure 4: Numerical estimate of the flow density dependence $\mathcal{K} \cdot V$ on car density $\mathcal{K}$ (Fundamental diagram).

Conclusion

1. Numerical results show the efficiency of transition to the basic integral equation and Markov chain simulation in VTF problems.

2. The transition presented here enables us to study parametric dependencies of functionals of our interest and apply various techniques to reduce computational costs.

Possible directions for future studies:

- more realistic interaction profiles (including random parameters);
- mixture of both driver behaviors and vehicle classes;
- multilane traffic with overtaking;
- cluster formation on the road;
- spatial inhomogeneity.

References

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[2] K.T. Waldeer. A vehicular traffic flow model based on a stochastic acceleration process // Transp. Theory Stat. Phys. **33** (1), 7–30 (2004)