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On the simulation of indistinguishable fermions in the many-body Wigner formalism

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ABSTRACT

The simulation of quantum systems consisting of interacting, indistinguishable fermions is an incredible mathematical problem which poses formidable numerical challenges. Many sophisticated methods addressing this problem are available which are based on the many-body Schrödinger formalism. Recently a Monte Carlo technique for the resolution of the many-body Wigner equation has been introduced and successfully applied to the simulation of distinguishable, spinless particles. This numerical approach presents several advantages over other methods. Indeed, it is based on an intuitive formalism in which quantum systems are described in terms of a quasi-distribution function, and highly scalable due to its Monte Carlo nature. In this work, we extend the many-body Wigner Monte Carlo method to the simulation of indistinguishable fermions. To this end, we first show how fermions are incorporated into the Wigner formalism. Then we demonstrate that the Pauli exclusion principle is intrinsic to the formalism. As a matter of fact, a numerical simulation of two strongly interacting fermions (electrons) is performed which clearly shows the appearance of a Fermi (or exchange–correlation) hole in the phase-space, a clear signature of the presence of the Pauli principle. To conclude, we simulate 4, 8 and 16 non-interacting fermions, isolated in a closed box, and show that, as the number of fermions increases, we gradually recover the Fermi–Dirac statistics, a clear proof of the reliability of our proposed method for the treatment of indistinguishable particles.

1. Introduction

The simulation of many-body quantum systems composed of indistinguishable fermions represents an incredible mathematical challenge. As a matter of fact, it is considered to be one of the most difficult problem in Computational Physics. At the same time, this is a problem of greatest importance. Indeed, our capability of simulating such particular quantum systems affects our life in many aspects. For example, such simulations allow the comprehension and design of innovative electronic devices, new and more effective drugs, materials with novel properties, all areas which have in common the resolution of the many-body quantum problem for fermions. It is, thus, not surprising that this problem has received a great deal of attention in the last decades.

Many methods are available which are based on different levels of approximation. The simplest one, known as the Hartree–Fock or self-consistent field method, [1,2] is based on expressing the many-body wave-function for fermions in terms of a Slater determinant (i.e. an antisymmetric wave-function) [3]. More sophisticated approaches exist among which we have the configuration interaction method [4], the Møller–Plesset method [5], etc. Another class of advanced

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post-Hartree–Fock methods is given by the quantum Monte Carlo methods which can handle multi-dimensional integrals, coming from the quantum many-body problem, by means of sophisticated Monte Carlo techniques [6–8]. All these mentioned methods rely on the ability of calculating stationary wave-functions of the many-body Schrödinger equation [9]. To the best of our knowledge, other formalisms of quantum mechanics such as the Wigner model [10], are not of common use in the simulation of quantum many-body problems.

Recently, a Monte Carlo (MC) method for the simulation of the single-particle Wigner equation has been introduced [11], based on (virtual) signed particles. This technique relies on the Iterative MC technique [12,13], a time-dependent approach which can deal with general initial and boundary conditions. It has been validated and applied with success to a number of technology relevant situations [11,14–17]. More recently, it has been generalized to the case of ab-initio calculations in the density functional theory (DFT) [18] and to the many-body Wigner equation involving distinguishable and spinless particles [19]. This generalization brings several important advantages in ab-initio calculations. Firstly, it is based on the Wigner model, equivalent to the Schrödinger equation, which describes a system in terms of the intuitive concept of a (quasi-)distribution function. Secondly, it is a MC technique and, as such, it is known to scale very well on parallel machines [13] (virtual particles are independent from each other and do not need any communication between nodes). Finally, it allows time-dependent simulations of distinguishable particles in ab-initio terms, opening the door to excited states calculations, etc.

In this paper, we extend the many-body Wigner MC method, described in [19], to the case of indistinguishable fermions. To this goal, we first show how to include the antisymmetric properties of a many-body wave-function in the Wigner formalism by following the approach suggested in [20]. Then, we perform two different numerical experiments to show the reliability of the method. These experiments are directly inspired by the works [21,22] where the Wigner formalism is investigated in the case of indistinguishable particles and where the mathematical expression [21] and behavior [22] of the Wigner quasi-distribution function is analyzed for the cases of interacting and non-interacting fermions. Accordingly, we start from two interacting (through their Coulomb potential) indistinguishable fermions (electrons) which travel in opposite directions with the same initial energy. A Fermi (or exchange–correlation) hole clearly appears in the phase-space, unveiling the presence of the Pauli exclusion principle in the simulation (as explained in [22]). Then we proceed with a relatively more complex situation involving non-interacting fermions isolated in a closed box (infinite potential barriers at the boundaries). By simulating 4, 8 and 16 particles, we recover the Fermi–Dirac distribution, a further clear proof of the reliability of the proposed method (as predicted in [22]).

2. The many-body Wigner Monte Carlo method for indistinguishable particles

In this section, we first sketch very briefly the method depicted in [19]. Then, we proceed with understanding how to include the antisymmetric property of a many-body wave-function in the Wigner formalism, basing our investigation on the analysis reported in [20]. Finally we provide some comments on the computational complexity of our MC approach.

2.1. The many-body Wigner Monte Carlo method

The many-body Wigner formulation of quantum mechanics [10] describes a system of n interacting particles in terms of a quasi-distribution function $f_W = f_W(\mathbf{x};\mathbf{p};t)$, with $(\mathbf{x};\mathbf{p};t) = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n;\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;t)$, $\mathbf{x}_i$ the position and $\mathbf{p}_i$ the momentum of the i-th particle ($i = 1,2,\ldots,n$) respectively. The Wigner function is expressed in terms of the solution of the many-body Schrödinger equation $\Psi = \Psi(\mathbf{x};t)$ via the invertible Wigner–Weyl transform which reads, in the case of pure state, as

$$f_W(\mathbf{x};\mathbf{p};t) = \frac{1}{(\hbar\pi)^{d\cdot n}} \int d\mathbf{x}' e^{-\frac{i}{\hbar}\sum_{k=1}^{n}\mathbf{x}'_k\cdot\mathbf{p}_k} \Psi\left(\mathbf{x}+\frac{\mathbf{x}'}{2};t\right)\Psi^*\left(\mathbf{x}-\frac{\mathbf{x}'}{2};t\right), \tag{1}$$

with $d = 1,2,3$ the dimensionality of space, $\int d\mathbf{x}' = \int d\mathbf{x}'_1 \int d\mathbf{x}'_2 \ldots \int d\mathbf{x}'_n$ and $(\mathbf{x}\pm\frac{\mathbf{x}'}{2}) = (\mathbf{x}_1\pm\frac{\mathbf{x}'_1}{2},\mathbf{x}_2\pm\frac{\mathbf{x}'_2}{2},\ldots,\mathbf{x}_n\pm\frac{\mathbf{x}'_n}{2};t)$. The Wigner–Weyl transform in the case of mixed states is, similarly, defined in terms of the many-body density matrix $\rho = \rho(\mathbf{x};\mathbf{y};t)$

$$f_W(\mathbf{x};\mathbf{p};t) = \frac{1}{(\hbar\pi)^{d\cdot n}} \int d\mathbf{x}' e^{-\frac{i}{\hbar}\sum_{k=1}^{n}\mathbf{x}'_k\cdot\mathbf{p}_k} \rho\left(\mathbf{x}+\frac{\mathbf{x}'}{2};\mathbf{x}-\frac{\mathbf{x}'}{2};t\right). \tag{2}$$

By applying the transform (1) to the many-body Schrödinger equation or, similarly, by applying the transform (2) to the many-body Liouville–von Neumann equation (see Eq. (5) below), one obtains the many-body Wigner equation:

$$\frac{\partial f_W}{\partial t}(\mathbf{x};\mathbf{p};t) + \sum_{k=1}^{n}\frac{\mathbf{p}_k}{m_k}\cdot\nabla_{\mathbf{x}_k} f_W = \int d\mathbf{p} f_W(\mathbf{x};\mathbf{p};t) V_W(\mathbf{x};\mathbf{p};t), \tag{3}$$

with $\int d\mathbf{p} = \int d\mathbf{p}_1 \int d\mathbf{p}_2 \ldots \int d\mathbf{p}_n$, m_i the mass of the i-th particle, and the Wigner kernel $V_W = V_W(\mathbf{x};\mathbf{p};t)$ defined as

$$V_W(\mathbf{x};\mathbf{p};t) = \frac{i}{\pi^{dn}\hbar^{dn+1}} \int d\mathbf{x}' e^{-(\frac{i}{\hbar})\sum_{k=1}^{n}\mathbf{x}'_k\cdot\mathbf{p}_k} \left[V\left(\mathbf{x}+\frac{\mathbf{x}'}{2};t\right) - V\left(\mathbf{x}+\frac{\mathbf{x}'}{2};t\right)\right]. \tag{4}$$

The function $V = V(\mathbf{x}; t)$ represents the potential acting on the involved particles, which can vary in time [19]. Thus, our problem consists of solving Eq. (3) in a specified domain along with given initial conditions (ICs) and boundary conditions (BCs). Please note that, so far, no use of the distinguishability (or not) of the particles is made in recovering Eq. (3).

By discretizing the space of momenta, rewriting the semi-discrete many-body Wigner equation as a Fredholm integral equation of second kind, and giving a physical interpretation of the terms of the Liouville–von Neumann series expressing the macroscopic variables of interest, it is possible to depict a MC method [19]. After any free flight, an initial virtual particle creates a pair of new virtual particles with opposite signs and momentum offset (around the initial momentum) equal to $+\mathbf{L}$ and $-\mathbf{L}$ with $\mathbf{L} = \mathbf{M} - \mathbf{M}'$. The parent particle and the child pair represent three contributive terms to the many-body Liouville–Neumann series [19]. This method is highly scalable, being the virtual signed particles completely independent from each other.

2.2. Indistinguishable particles in the Wigner formalism

In order to proceed with the inclusion of indistinguishable fermions in the many-body Wigner formalism, we follow the reasoning of [20] and start from the Liouville–von Neumann equation which, in the Schrödinger formalism, reads

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H, \rho], \tag{5}$$

where H is the Hamiltonian of the system, which includes the kinetic operator and the (time-dependent) potential, and $[H, \rho]$ is the commutator operator $H\rho - \rho H$. One can show that, by applying the transform (2) to the many-body Liouville–von Neumann equation (5), it is possible to recover the many-body Wigner equation (3). Please note that no assumption is made on the symmetry properties of the system yet [20].

Now, indistinguishable fermions in the Schrödinger formalism are described by antisymmetric wave-functions, i.e.

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n; t) = -\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_n; t). \tag{6}$$

Taking into account the nature of the system, one can define a Wigner–Weyl transform for fermions

$$f_W^-(\mathbf{x}; \mathbf{p}; t) = \frac{1}{(\hbar\pi)^{d\cdot n}} \int d\mathbf{x}' e^{-\frac{i}{\hbar}\sum_{k=1}^{n} \mathbf{x}'_k \cdot \mathbf{p}_k} \Psi^- \left(\mathbf{x}_1 + \frac{\mathbf{x}'_1}{2}, \ldots, \mathbf{x}_n + \frac{\mathbf{x}'_n}{2}; t\right) \Psi^{-*} \left(\mathbf{x}_1 - \frac{\mathbf{x}'_1}{2}, \ldots, \mathbf{x}_n - \frac{\mathbf{x}'_n}{2}; t\right), \tag{7}$$

where $\Psi^-(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is an antisymmetric many-body wave-function (the case for mixed states is treated similarly in [20]). By applying the transform (7) to Eq. (5), one recovers a many-body Wigner equation for f_W^- valid for fermions which is mathematically equivalent to (3):

$$\frac{\partial f_W^-}{\partial t}(\mathbf{x}; \mathbf{p}; t) + \sum_{k=1}^{n} \frac{\mathbf{p}_k}{m} \cdot \nabla_{\mathbf{x}_k} f_W^- = \int d\mathbf{p} f_W^-(\mathbf{x}; \mathbf{p}; t) V_W(\mathbf{x}; \mathbf{p}; t). \tag{8}$$

This proves, in a clear manner, that the whole Wigner formalism does not need any change to handle the case of antisymmetric systems [20]. In particular, for a system of fermions, the Pauli exclusion principle seems to be directly embedded into this formalism and does not need to be, somehow, imposed during the simulation.

Thus, the many-body Wigner equation for distinguishable particles (3) is also valid for the case of indistinguishable fermions (8), and the antisymmetric properties of the system are imposed through the ICs only, by using the transform (7). As a consequence, in order to handle systems of fermions, one can start from a Slater determinant

$$\Psi^-(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \ldots & \phi_n(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \ldots & \phi_n(\mathbf{x}_2) \\ \ldots & \ldots & \ldots & \ldots \\ \phi_1(\mathbf{x}_n) & \phi_2(\mathbf{x}_n) & \ldots & \phi_n(\mathbf{x}_n) \end{vmatrix}. \tag{9}$$

It is possible to prove that this is equivalent to mathematically express the initial Wigner function as a sum of reduced single-particle Wigner functions (one for each body involved) plus a certain number of integral terms [21,22]. This introduces a certain degree of coupling between the involved fermions, in agreement with the fact that the corresponding initial Wigner function cannot be expressed as multiplications of independent wave-packets only.

2.3. Computational complexity

Finally, we would like to comment on the computational complexity of the many-body Wigner MC method. It is relatively easy to demonstrate that the complexity of the MC part dealing only with the evolution of the phase-space coordinates of the virtual particles increases linearly with the number of bodies involved [13]. Unfortunately, the same cannot be said about the calculation of the Wigner kernel (4). Indeed, this is equivalent to the calculation of a function defined over a space which dimensions increase exponentially with the number of bodies. In other words, the Wigner kernel is, in the time-dependent many-body formalism, the new bottleneck of the quantum many-body problem. While this does not

represent a problem for non-interacting fermions (where the function $\gamma = \gamma(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is time-independent, even in the case of entangled particles), it is a severe limitation when all interactions have to be taken into account in a consistent way (the Coulombic interactions have to be updated at every time step).

3. Numerical simulations of fermions

In this section, we perform two different numerical experiments involving interacting and non-interacting indistinguishable fermions. The goal of these simulations is twofold. Firstly, we show that the Pauli exclusion principle is actually taking place into the evolution of fermions in the many-body Wigner formalism, without the need to impose it. In other words, it is embedded in the formalism as soon as the initial conditions correspond to a system of fermions (i.e. the initial Wigner function is calculated from an antisymmetric wave-function). Secondly, we show that the many-body Wigner MC method can handle actual many-body problems such as a gas of non-interacting fermions in a closed box. By simulating 4, 8 and 16 electrons we gradually recover the Fermi–Dirac distribution, a clear proof of the reliability of the method. The results presented in this work follow the ideas reported in [21] and [22] where the Wigner formalism is investigated for the case of interacting and non-interacting indistinguishable fermions. In particular, in [21] the mathematical expression of the Wigner quasi-distribution function is analyzed in terms of single-particle distribution functions, while in [22] the time-dependent behavior of the Wigner function is studied and numerical results are reported. A simple comparison shows a good agreement between the two approaches despite their differences (Wigner paths method vs. signed particle many-body Wigner MC method).

3.1. The Pauli exclusion principle

The system simulated in this first numerical experiment consists of two indistinguishable fermions, trapped in a one-dimensional box, interacting with each other through their generated Coulombic potential. The system starts from the initial conditions

$$f_W^0(x_1, x_2; p_1, p_2) = \frac{1}{(\hbar\pi)^2} \int dx_1' dx_2' e^{-\frac{i}{\hbar}(x_1' p_1 + x_2' p_2)} \Psi_0\left(x_1 + \frac{x_1'}{2}, x_2 + \frac{x_2'}{2}\right) \Psi_0^*\left(x_1 - \frac{x_1'}{2}, x_2 - \frac{x_2'}{2}\right),$$

with

$$\Psi_0(x_1, x_2) = \begin{vmatrix} \phi_1(x_1) & \phi_2(x_2) \\ \phi_1(x_2) & \phi_2(x_2) \end{vmatrix},$$

(Slater determinant, i.e. antisymmetric wave-function) and

$$\phi_1(x) = N_1 e^{-\frac{1}{2}(\frac{x-x_1^0}{\sigma})^2} e^{ip_1^0 x},$$

$$\phi_2(x) = N_2 e^{-\frac{1}{2}(\frac{x-x_2^0}{\sigma})^2} e^{ip_2^0 x},$$

where N_1 and N_2 are normalization constants, x_1^0 and x_2^0 are the initial central positions, p_1^0 and p_2^0 are the initial momenta, for the first and the second wave-packet respectively, and σ the initial dispersion of the wave-packets. In this particular experiment, the values $x_1^0 = 20$ nm, $x_2^0 = 30$ nm, $p_1^0 = +2\Delta p$, $p_2^0 = -2\Delta p$, and $\Delta p = \frac{\pi}{\hbar L_C}$ with $L_C = 20$ nm, have been used and the evolution is performed until time 3.5 fs. Finally, the total length of the spatial domain is equal to 150 nm. The results of the simulations are reported in Figs. 1, 2, 3 and 4. In particular Figs. 1 and 2 show the evolution of the quantity $\int d\mathbf{x}_1 d\mathbf{p}_1 f_W(\mathbf{x}_1, \mathbf{x}_2; \mathbf{p}_1, \mathbf{p}_2; t) + \int d\mathbf{x}_2 d\mathbf{p}_2 f_W(\mathbf{x}_1, \mathbf{x}_2; \mathbf{p}_1, \mathbf{p}_2; t)$ in the phase-space at time 0 fs, 1 fs, 2.5 fs and 3.5 fs respectively, while Figs. 3 and 4 show the evolution of the corresponding probability density in the spatial domain.

The two particles start with the same amount of energy but with opposite momenta, Fig. 1 (top). They proceed one from right to left, the other from left to right feeling their repulsive Coulombic potential, Fig. 1 (bottom). At time 2.5 fs, Fig. 2 (top), a Fermi (or exchange–correlation) hole is clearly visible. Intuitively speaking, this hole appears because of the strong electric interactions happening between the two (indistinguishable) fermions, in other words they repel each other. Indeed, this experiment is equivalent to *pinching* two electrons with same energy against each other and, accordingly, a lower probability is developed in the central area of the phase-space, preventing the two particles to be in the same position with the same energy. This is a proof of the presence of the Pauli exclusion principle during the simulation [21,22] which, essentially, states that two particles with the same spin cannot be in the same orbital at the same time. Eventually, as the evolution proceeds in time, the Fermi hole disappears, Fig. 2 (bottom) [22].

3.2. The Fermi–Dirac statistics

We now proceed with the simulation of n non-interacting fermions, with $n = 4, 8, 16$, confined in a closed box. It is well known that such a system, for a long enough time, obeys the Fermi–Dirac distribution. Thus, this can be considered as a further benchmark test to evaluate the reliability of our proposed MC method. The results, for the case $n = 4, 8, 16$, are reported in Fig. 5 where a comparison is made against the Fermi–Dirac distribution. The system temperature T is equal to

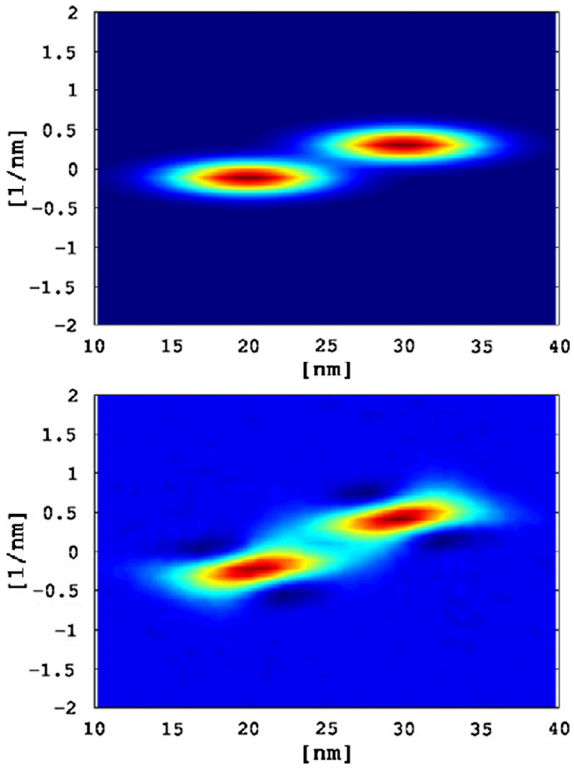

Fig. 1. Time-dependent evolution of the reduced one-particle Wigner function [22] at time 0 fs (top) and 1 fs (bottom). The x- and y-axes refer to position and momentum respectively.

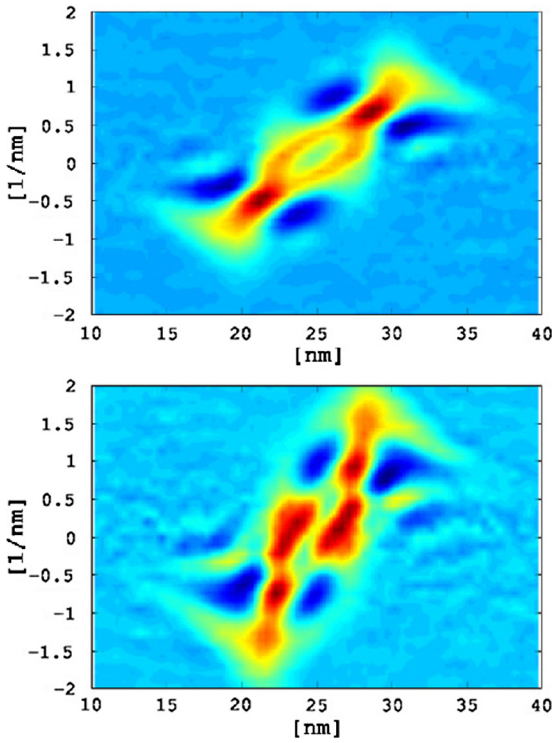

Fig. 2. Time-dependent evolution of the reduced one-particle Wigner function [22] at time 2.5 fs (top) and 3.5 fs (bottom). The formation of a Fermi hole (also known as an exchange–correlation hole), due to the Pauli exclusion principle, is clearly visible at time 2.5 fs (top). Eventually the hole disappears (3.5 fs) as the system evolves (bottom). The x- and y-axes refer to position and momentum respectively.

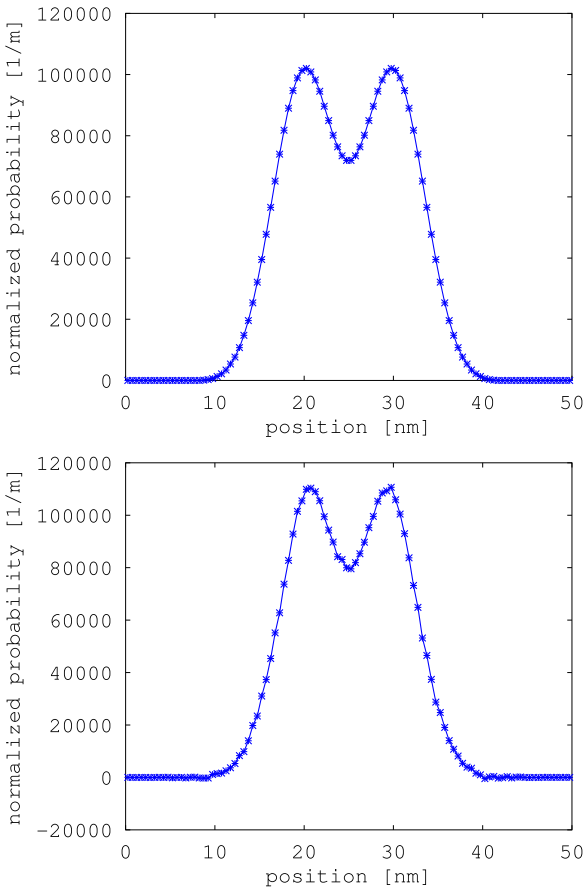

Fig. 3. Time-dependent evolution of the 2-Fermion density at time 0 fs (top) and 1 fs (bottom) corresponding to the experiment performed in Fig. 1.

2 K and the evolution is performed until time 500 fs. The domain length L_x is 150 nm long and has two infinite potential walls at the boundaries. The initial conditions consist, again, of a Wigner–Weyl transform of an antisymmetric n-body Slater determinant, and it is possible to show that the initial Wigner function can be expressed as the sum of reduced single-body Wigner distribution functions and other extra coupling terms [21,22]. Finally, the chemical potential μ for the various experiments is calculated from the following formula

$$n = \sum_{k=1}^{+\infty} \frac{1}{e^{\frac{\epsilon_k - \mu}{k_B T}} + 1},$$

where $n = 4, 8, 16$, $\epsilon_k = \frac{k^2 \hbar^2 \pi^2}{2mL_x^2}$, m is the mass of an electron, and k_B is the Boltzmann constant. For every choice of n, one evolves the system until the final time (500 fs). Thus, the quantity $f_W(\epsilon)$, with $\epsilon = \frac{p^2}{2m}$, is calculated.

The results obtained by means of the many-body Wigner MC method for 4, 8, 16 fermions are compared to the Fermi–Dirac distribution function in Fig. 5. Clearly, the agreement improves as the number of fermions increases. This behavior is similar to the one reported in [22]. This shows that our method is reliable and can be utilized for the simulation of systems of indistinguishable fermions.

3.3. Software and hardware

The results presented in this section have been obtained using the HPC cluster installed at the Institute of Information and Communication Technologies (IICT) of the Bulgarian Academy of Sciences. This cluster consists of two racks which contain HP Cluster Platform Express 7000 enclosures with 36 blades BL 280c with dual Intel Xeon X5560 @ 2.8 GHz (total 576 cores), 24 GB RAM per blade. There are 8 storage and management controlling nodes 8 HP DL 380 G6 with dual Intel X5560 @ 2.8 GHz and 32 GB RAM. All these servers are interconnected via non-blocking DDR Infiniband interconnect at 20 Gbps line speed. The theoretical peak performance is 3.23 Tflops.

The simulator used to obtain the results presented in this paper is a modified version of Archimedes, the GNU package for the simulation of carrier transport in semiconductor devices [23]. This code was first released in 2005, and, since then,

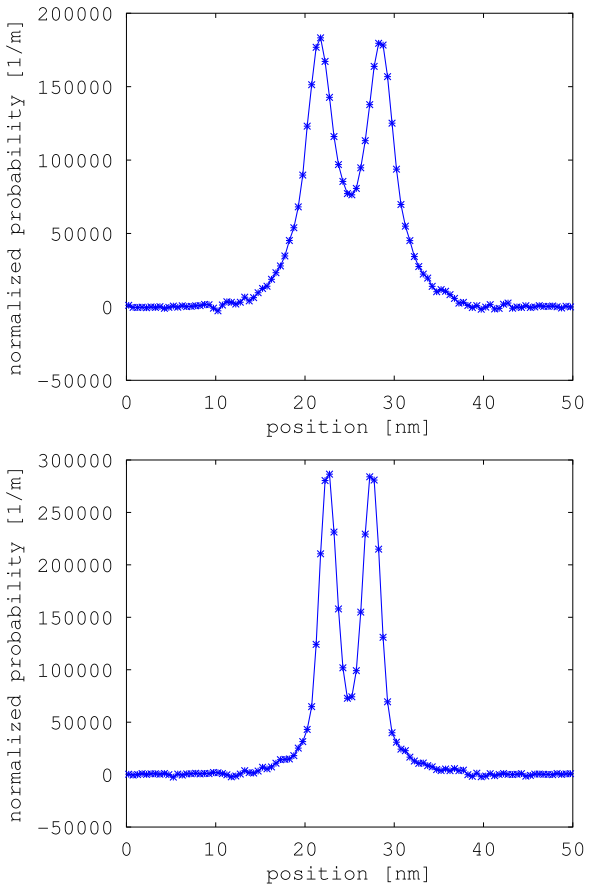

Fig. 4. Time-dependent evolution of the 2-Fermion density at time 2.5 fs (top) and 3.5 fs (bottom) corresponding to the experiment performed in Fig. 2.

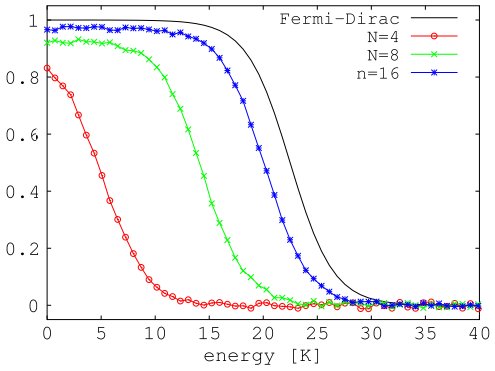

Fig. 5. Energy-dependent distribution $f_W(\epsilon)$ [22]. The solution is compared to the Fermi–Dirac distribution for $n = 4, 8, 16$ non-interacting fermions confined between two infinite potential barriers. The domain length is equal to 150 nm, the temperature is equal to 2 K and the particles are evolved until a time 200 fs.

users have been able to download the source code under the GNU Public License (GPL). Many features have been introduced in this package. In this particular project, our aim is to develop a full quantum simulator. The code is entirely developed in C and optimized to get the best performance from the hardware. The results of the new version will be posted on the nano-archimedes website (see [24]).

4. Conclusions

In this work, we extended the many-body Wigner Monte Carlo method to the case of indistinguishable particles with spin. In particular, we focused on the simulation of fermions, a problem known to be numerically difficult. We have shown

how the Wigner formalism includes the treatment of fermions in a very simple and natural way, demonstrating that practically the evolution equation remains the same for spinless distinguishable particles and for indistinguishable particles with spin [20]. What actually changes is the way the initial conditions must be imposed. Indeed, for example, in the case of fermions, the antisymmetric properties of the many-body wave-function have to be included in the calculation of the initial Wigner quasi-distribution function by means of the Wigner–Weyl transform (1) applied to a Slater determinant (9). By performing two different numerical experiments we have shown that, as soon as the proper initial conditions are enforced, the Pauli exclusion principle appears in the form of a Fermi (or exchange–correlation) hole, see Fig. 2 (top). We also have shown how our proposed method can handle more realistic, and thus more computationally demanding, problems such as the simulation of a gas of non-interacting fermions isolated in a closed box. As a matter of fact, as the number of fermions increases, we recover the Fermi–Dirac distribution, a clear sign that the many-body Wigner Monte Carlo approach works properly even in the case of indistinguishable particles. This opens the way toward affordable time-dependent simulations of quantum systems constituted of identical fermions.

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