Monte Carlo Estimator for Image Creation with Symmetric Sampling of Phong BRDF Model

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Abstract. The paper is directed to the advanced rendering techniques for realistic image creation. We construct and offer a new Monte Carlo estimator for numerical solution of the rendering equation based on Phong BRDF (Bidirectional Reflectance Distribution Function) model. We consider the kernel of Phong rendering equation and present the Monte Carlo solution by a sum of two independent integrals, one for diffuse and one for specular part respectively. The diffuse integral equation is solved by applying Combined Uniform Separation of integration domain to achieve variance reduction. The hemispherical integration domain is symmetrically separated into 8 sub-domains of equal orthogonal spherical triangles and 8 sub-domains of equal spherical quadrangles. All spherical triangles, spherical quadrangles respectively are symmetric each to other as well as have fixed vertices and computable parameters. The symmetric sampling scheme is applied to generate the sampling points and solve the diffuse integral equation. The integration domain of specular integral equation is approximated by conical solid angle of most important region of interest. The normal vector of the conical solid angle is the direction of ideally reflected by the surface viewing vector. We show that the conical solid angle can be successfully approximated with rotation of all orthogonal spherical triangles sub-domains, constructed at solving the diffuse integral equation, being easy to reuse the sampling points.

Keywords: Monte Carlo, Uniform Separation, Rendering Equation, Phong BRDF model, Image Synthesis, Fredholm Integral Equations.

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INTRODUCTION

The Monte Carlo methods are proved to be the only effective method for photo-realistic image creation [1], where the global illumination problem must to be solved numerically. The global illumination problem considers the light transport in a closed virtual scene. From mathematical point of view, the solution of global illumination problem is equivalent to the solution of the rendering equation, which is a second kind Fredholm type integral equation [2] and describes the light propagation in a scene (see in Fig. 1(a)). The radiance L, leaving from a point x on the surface of the scene in direction $\omega \in \Omega_x$, where Ω_x is the hemisphere at point x, is the sum of the self radiating light source radiance L^e and all reflected radiance: $L(x,\omega) = L^e(x,\omega) + \int_{\Omega_x} L(h(x,\omega'),-\omega') f_r(-\omega',x,\omega)\cos\theta' d\omega'$, where $h(x,\omega')$ is the first point that is hit when shooting a ray from x into direction ω'. The radiance L^e has non-zero

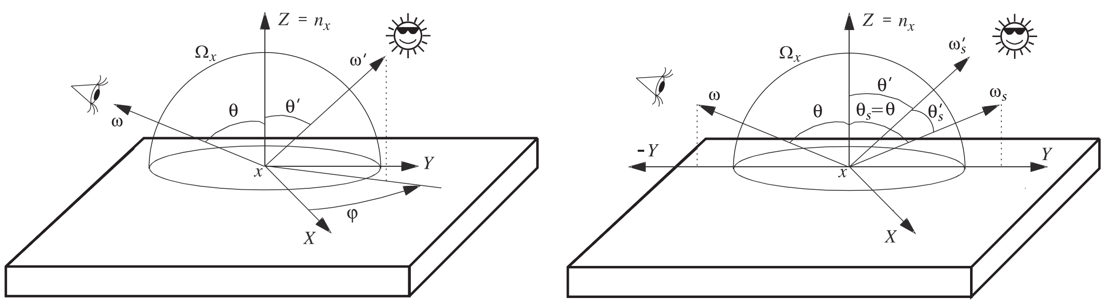

FIGURE 1. The geometry for **(a)** diffuse part **(b)** and specular part of the rendering equation

value if the considered point x is a point from solid light source. Therefore, the reflected radiance in direction ω is an integral of the radiance incoming from all points, which can be seen through the hemisphere Ω_x at point x attenuated by the surface BRDF (Bidirectional Reflectance Distribution Function) $f_r(-\omega', x, \omega)$ and the projection $\cos\theta'$. The angle θ' is the angle between the surface normal n_x at point x and the direction ω'. The law for energy conservation holds, i.e.: $\int_{\Omega_x} f_r(-\omega', x, \omega)\cos\theta' d\omega' < 1$, because a real scene always reflects less light than it receives from the light sources due to light absorption of the objects.

The kernel of the rendering equation is determined by applied BRDF model. Several BRDF models are designed for Monte Carlo solutions of the rendering equation. One useful and frequently used in image creation is Phong BRDF model, which we consider further in the paper to construct our Monte Carlo estimator. We apply the Uniform Separation Strategy [3] in order to achieve variance reduction and improved the efficiency of Monte Carlo estimator.

INTEGRATION DOMAIN APPROXIMATION FOR PHONG BRDF MODEL

Usually, the BRDF is modeled like a sum of two independent components: diffuse and specular. The diffuse part integrates all the light incoming from all directions $-\omega'$ and reflected in direction ω as shown in Fig. 1(a). The specular component (see in Fig. 1(b)) accounts for the light incoming from and around a particular direction $-\omega_s$, which is ideal-mirror reflected direction to ω. Note that the directions ω, $-\omega_s$ and the normal vector $\overrightarrow{n}_x = \overrightarrow{Z}$ are in the plane $(\overrightarrow{Y}, \overrightarrow{Z})$ of a Descartes coordinate system or with simple rotation on a certain angle φ to fulfil this condition.

In Phong BRDF model $f_r(-\omega', x, \omega) = \rho_d + \rho_s \cos^n \theta'_s$, where ρ_d and ρ_s are constants, $\rho_d + \rho_s \leq 1$ preserves the energy conservation law and θ'_s is the angle between ω_s and ω'_s directions. Therefore the rendering equation with Phong BRDF is

$$L(x,\omega) = L^e(x,\omega) + \int_{\Omega_x} L(h(x,\omega'), -\omega')(\rho_d + \rho_s \cos^n \theta'_s)\cos\theta' d\omega', \tag{1}$$

and let for simplification we assume that $L^e(x,\omega) = 0$ for ***Eq. (1)*** we obtain $L(x,\omega) = L_1 + L_2$, where

$$L_1 = \int_{\Omega_x} L(h(x,\omega'), -\omega')\rho_d \cos\theta' d\omega' \qquad \text{and} \qquad L_2 = \int_{\Omega_x} L(h(x,\omega'_s), -\omega'_s)\rho_s \cos^n \theta'_s \cos\theta' d\omega'.$$

Since we have two independent integrals we can solve them separately with Monte Carlo methods. Consider the kernel of integral L_2, one can see that $\cos^n \theta'_s$ term has very poor or negligible contribution for big values of θ'_s or when $\theta'_s \to \frac{\pi}{2}$ respectively. Practically, only a part of the light incoming at point x from a solid angle surrounding the direction $-\omega'_s$ takes the influence at specular component computations. This fact gives us the possibility to approximate the integration domain Ω_x with a conical solid angle Ω_s centered around ω_s direction, which is of most important computational interest. Let for simplification denote $L(h(x,\omega'_s), -\omega'_s) = L_{\Omega_s}$, we can approximate the integral L_2 as:

$$L_2 = \int_{\Omega_x} L(h(x,\omega'_s), -\omega'_s)\rho_s \cos^n \theta'_s \cos\theta' d\omega' \approx \int_{\Omega_s} L_{\Omega_s}\rho_s \cos^n \theta'_s \cos\theta' d\omega'. \tag{2}$$

MONTE CARLO ESTIMATOR

We shall use symmetric sampling to construct and design our Monte Carlo estimator. The symmetric sampling points are generated according to the Uniform Separation of integration domain strategy. Uniform Separation strategy is introduced for first time by us in [3], further developed and extended in [4]. The variance reduction achieved with Uniform Separation is considered, analyzed and proved in [5], [6] and [7].

Let us consider the diffuse part of ***Eq. (1)*** for which we apply Combined Uniform Separation of integration domain Ω_x in a manner described in [4] and shown in Fig.2(a). The hemispherical integration domain Ω_x is symmetrically separated into 16 parts, $\Omega_x = \bigcup_{i=0}^{i=7}\left(\Omega_{\triangle_i} \bigcup \Omega_{\square_i}\right)$ where for each i, j : $\Omega_{\triangle_i} \cap \Omega_{\square_j} = \Omega_{\triangle_i} \cap \Omega_{\triangle_j} = \Omega_{\square_j} \cap \Omega_{\square_i} \varnothing$. First 8 sub-domains are equal size of orthogonal spherical triangle $\Omega_{\triangle_i}$. They are symmetric each to other and grouped with a common vertex around the normal vector to the surface. The hemispherical integration domain is completed with more 8 sub-domains of equal size spherical quadrangle $\Omega_{\square_i}$, also symmetric each to other. All sub-domains have fixed vertices and computable parameters. The bijections $F(u,v) : [0,1]^2 \to \Omega_\square$ and $F(u,v) : [0,1]^2 \to \Omega_\triangle$ map the uniformly distributed in $[0,1]$ random variables u and v into the sampling points on the spherical quadrangle $\Omega_\square$ and on orthogonal spherical triangle $\Omega_\triangle$, respectively. The symmetric property allows us to calculate in parallel the coordinates of the symmetric points. Once a sampling point $P_0(X_0, Y_0, Z_0)$ is generated in $\Omega_\triangle$ or $\Omega_\square$, the coordinates of all symmetric point from the other sub-domains can be obtained by simple change of the signs and/or places of X_0 and Y_0, while Z_0 is not affected. For example, $P_1(Y_0, X_0, Z_0)$, $P_2(Y_0, -X_0, Z_0)$, $\cdots$ and $P_7(X_0, -Y_0, Z_0)$, same at $\bar{P}$ points.

We can solve (see [3]) the orthogonal spherical triangle $\Omega_{\triangle_0}$ and the spherical quadrangle $\Omega_{\square_0}$ in Fig. 2(a) to derive the necessary transformation. We obtain for both bijections $F(u,v):[0,1]^2 \to \Omega_{\triangle_0}$ and $F(u,v):[0,1]^2 \to \Omega_{\square_0}$ the transformations: $\left(\varphi_0 = \frac{u\pi}{4}; \theta'_{\triangle_0} = \arctan\frac{v}{\cos\varphi_0} = \arctan\frac{v}{\cos\frac{u\pi}{4}}\right)$ and $\left(\varphi_0 = \frac{u\pi}{4}; \theta'_{\square_0} = \text{arccot}\frac{v}{\cos\varphi_0} = \text{arccot}\frac{v}{\cos\frac{u\pi}{4}}\right)$, where $u,v \in [0,1]$; $\varphi_0 \in [0,\frac{\pi}{4}]$ and $\theta'_{\triangle_0} \in [0, \arctan\frac{1}{\cos\varphi_0}]$ and $\theta'_{\square_0} \in [\arctan\frac{1}{\cos\varphi_0}, \frac{\pi}{2}]$. Consider the equation

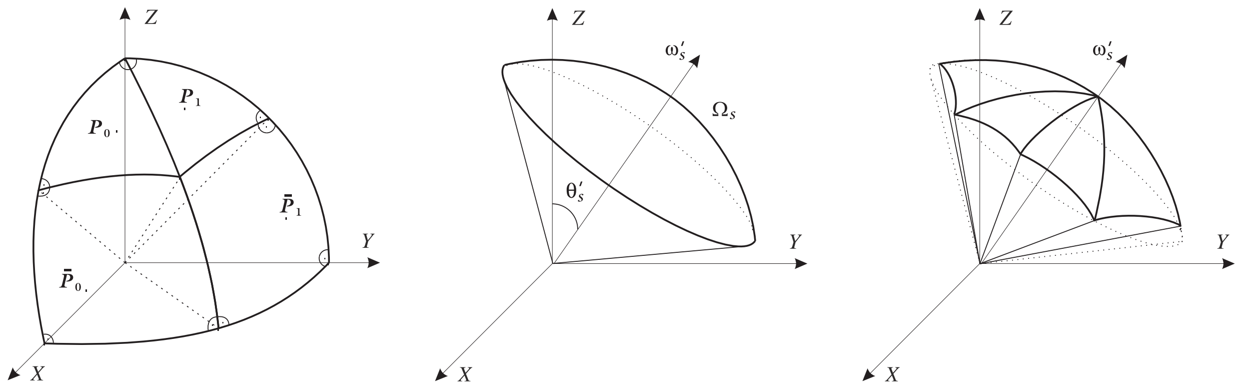

FIGURE 2. **(a)** Ω_x partition into $\Omega_\triangle$ and $\Omega_\square$ **(b)** conical solid angle Ω_s **(c)** approximation of Ω_s with $\Omega_\triangle$

$L_1 = \int_{\Omega_x} L(h(x,\omega'), -\omega')\rho_d \cos\theta' d\omega'$, applying the partition of Ω_x into non-overlapping sub-domains $\Omega_{\triangle_i}$ and $\Omega_{\square_i}$ for $i = 0,1,\ldots,7$. It can be rewritten as:

$$L_1 = \rho_d \sum_{i=0}^{7} \left(\int_{\Omega_{\triangle_i}} L_{\triangle_i} \cos\theta'_{\triangle_i} d\omega'_{\triangle_i} + \int_{\Omega_{\square_i}} L_{\square_i} \cos\theta'_{\square_i} d\omega'_{\square_i} \right), \tag{3}$$

where the symbols $\triangle_i$ and $\square_i$ indicate the relevant functions or directions in $\Omega_{\triangle_i} = \frac{1}{24}\Omega_x = \frac{\pi}{12}$ and $\Omega_{\square_i} = \frac{1}{12}\Omega_x = \frac{\pi}{6}$ for $i = 0,1,\ldots,7$. We can transform ***Eq. (3)*** by the bijections $F(u,v):[0,1]^2 \to \Omega_\triangle$ and $F(u,v):[0,1]^2 \to \Omega_\square$ in the following form $L_1 = \frac{\rho_d\pi}{4}\sum_{i=0}^{7}\int_0^1\int_0^1 \left(L_{\triangle_i} + L_{\square_i}\right)\left(\frac{v\cos^2\frac{u\pi}{4}}{\left(v^2+\cos^2\frac{u\pi}{4}\right)^2}\right) du dv$.

We generate N' independent sampling points $Q_j(u_j,v_j) \in [0,1]^2$, $j = 1,2,\ldots,N'$ and sample each sub-domain by N' random points with a density function $p(\omega')/p_{\triangle_i}$ in $\Omega_{\triangle_i}$ or $p(\omega')/p_{\square_i}$ in $\Omega_{\square_i}$. One can see [4] the probability $p = \int_{\Omega_x} p(\omega')d\omega' = \sum_{i=0}^{7}\left(\int_{\Omega_{\triangle_i}} p(\omega')d\omega' + \int_{\Omega_{\square_i}} p(\omega')d\omega'\right) = \sum_{i=0}^{7}\left(p_{\triangle_i} + p_{\square_i}\right) = 1$ and $p_{\triangle_i} = \int_{\Omega_{\triangle_i}} p(\omega')d\omega' = \frac{1}{24}$ and $p_{\square_i} = \int_{\Omega_{\square_i}} p(\omega')d\omega' = \frac{1}{12}$ for $i = 0,1,\ldots,7$. We obtain the Monte Carlo $\langle L_1\rangle$ estimator for ***Eq. (3)*** as $\langle L_1\rangle \approx L_1$,

$$\langle L_1\rangle = \frac{\rho_d\pi}{4}\sum_{i=0}^{7}\frac{1}{N'}\sum_{j=0}^{N'}\left(p_{\triangle_i}L_{\triangle_i} + p_{\square_i}L_{\square_i}\right)\frac{v_j\cos^2\frac{u_j\pi}{4}}{(v_j^2+\cos^2\frac{u_j\pi}{4})^2} = \frac{\rho_d\pi}{48N'}\sum_{i=0}^{7}\sum_{j=0}^{N'}\left(\frac{1}{2}L_{\triangle_i} + L_{\square_i}\right)\frac{v_j\cos^2\frac{u_j\pi}{4}}{(v_j^2+\cos^2\frac{u_j\pi}{4})^2}. \tag{4}$$

Similarly, we consider ***Eq. (2)*** to be numerically solved in a conical solid angle Ω_s centered around ω_s direction (see in Fig.2(b)). We sample ***Eq. (2)*** with a probability density function proportional to the specular lobe of Phong BRDF model (see [8]) $p(\omega') = \frac{cos^n\theta'_s}{\int_{\Omega_s}\cos^n\theta'_s d\omega'_s}$, where $\theta'_s \in [0, \theta'_{s,max}]$ and $\int_{\Omega_s}\cos^n\theta'_s d\omega'_s = \frac{2\pi(1-cos^{n+1}\theta'_{s,max})}{(n+1)}$. Therefore ***Eq.(2)*** is

$$L_2 = \rho_s\left(\int_{\Omega_s}\cos^n\theta'_s d\omega'_s\right)\int_{\Omega_s} L_{\Omega_s}\cos\theta' d\omega' = \frac{\rho_s 2\pi(1-cos^{n+1}\theta'_{s,max})}{(n+1)}\int_{\Omega_s} L_{\Omega_s}\cos\theta' d\omega'. \tag{5}$$

One can see in Fig. 2(c) that the domain Ω_s is very similar by form to the domain $\Omega_\triangle = \bigcup_{i=0}^{i=7}\Omega_{\triangle_i}$. We can approximate Ω_s by rotating on angle θ_s domain $\Omega_\triangle = \bigcup_{i=0}^{i=7}\Omega_{\triangle_i}$, due to ***the directions ω, $-\omega_s$ and the normal vector $\overrightarrow{n}_x = \overrightarrow{Z}$ are in the plane $(\overrightarrow{Y}, \overrightarrow{Z})$ condition*** (see in Fig. 1(b) and Fig. 2(c)), where $\Omega_s \approx \Omega_s^\triangle = \bigcup_{i=0}^{i=7}\Omega_s^{\triangle_i}$ and $cos\theta_{s,max} = \frac{1}{\sqrt{3}}$. This is equivalent to applying the Uniform Triangle Separation of integration domain $\Omega_s \approx \Omega_s^\triangle$ and ***Eq. (5)*** is rewritten as:

$$L_2 \approx \frac{\rho_s 2\pi(1-cos^{n+1}\theta'_{s,max})}{(n+1)}\int_{\Omega_s^\triangle} L_{\Omega_s^\triangle}\cos\theta' d\omega' = \frac{\rho_s 2\pi}{n+1}\left(1-\left(\frac{\sqrt{3}}{3}\right)^{n+1}\right)\sum_{i=0}^{i=7}\int_{\Omega_s^{\triangle_i}} L_{\Omega_s^{\triangle_i}}\cos\theta'_{\triangle_i} d\omega'. \tag{6}$$

We ***reuse*** already generated sampling points in $\Omega_\triangle$ to find the sampling directions in $\Omega_s^\triangle$. Each sampling point $P(X,Y,Z) \in \Omega_\triangle$ is rotated on angle θ_s around $\overrightarrow{X}$ to obtain the sampling direction $P'(X',Y',Z') \in \Omega_s^\triangle$ by the rotation:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R_{\overrightarrow{X}}(\theta_s) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_s & -\sin\theta_s \\ 0 & \sin\theta_s & \cos\theta_s \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y\cos\theta_s - Z\sin\theta_s \\ Z\cos\theta_s + Y\sin\theta_s \end{bmatrix},$$

where $R_{\overrightarrow{X}}(\theta_s)$ is the rotation matrix. This rotation also allows us to calculate at once $Z' = Z\cos\theta_s + Y\sin\theta_s$ term or $Z' = \cos\theta\cos\theta_s + \sin\phi\sin\theta\sin\theta_s = \frac{\cos(\frac{u\pi}{4})}{\sqrt{\cos^2(\frac{u\pi}{4})+v^2}}\cos\theta_s + \frac{v}{\sqrt{\cos^2(\frac{u\pi}{4})+v^2}}\sin\frac{u\pi}{4}\sin\theta_s$ after applying the bijections $F(u,v) : [0,1]^2 \to \Omega_{\triangle_0}$. Finally, for N' independent sampling points $Q_j(u_j,v_j) \in [0,1]^2$, $j = 1,2,\ldots,N'$, we obtain $N = 8N'$ sampling points in $\Omega_s^\triangle$. In general case $\cos\theta'_{\triangle_{i,j}} = Z'_{i,j}(u_j,v_j)$ in ***Eq. (6)***. The Monte Carlo estimator for Phong BRDF specular component is $\langle L_2 \rangle \approx L_2$ or

$$\langle L_2 \rangle = \frac{2\pi\rho_s}{N(n+1)} \left(1 - \left(\frac{\sqrt{3}}{3}\right)^{n+1}\right) \sum_{i=0}^{i=7} \sum_{j=1}^{N'} L_{\Omega_{s,j}^{\triangle_i}} Z'_{i,j}(u_j,v_j). \tag{7}$$

The new Monte Carlo estimator for Phong BRDF model is the sum of ***Eq.(4)*** and ***Eq. (7)***

$$\langle L \rangle = \langle L_1 \rangle + \langle L_2 \rangle. \tag{8}$$

CONCLUSION

A new Monte Carlo estimator for solving Phong rendering equation is proposed. Our estimator offers variance reduction due to using of Uniform Separation Strategy. One can see, $\int_{\Omega_s} \cos^n\theta'_s d\omega'_s$ is difficult to be solved analytically, a more accurate estimation could be found as $\int_{\Omega_s} \cos^n\theta'_s d\omega'_s \approx \frac{2\pi}{(n+1)}\left(1 - \frac{1}{2}\left[\left(\frac{\sqrt{3}}{3}\right)^{n+1} + \left(\frac{\sqrt{2}}{2}\right)^{n+1}\right]\right)$. In our estimator the sampling points are reused in a natural way by symmetry and rotation, which are very fast computing operations. This is another important advantage, improving the efficiency of Monte Carlo estimator. The new Monte Carlo estimator ***Eq. (8)*** is easy for parallel realizations as well as implementations on GPU (Graphics Processing Unit).

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