Examining the Distribution of Sampling Point Sets on Sphere for Monte Carlo Image Rendering

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Abstract. This paper presents a study of several non-uniform and uniform techniques for sampling of unit sphere and gives an comparative analysis on uniform sampling techniques. Each sampling technique generates point sets distributed on sphere. We are interested to examine the quality of their distributions, which could impact on the Monte Carlo image creation. Generalized discrepancy is designed as a measure for equidistribution of point sets on spherical sampling patterns as well as applied to analyze sampling techniques used in rendering. We generate sphere sampling patterns using various random number generators and Halton low discrepancy sequence. The generalized discrepancy is computed for increasing number of points for each sampling pattern. Finally, the results for different uniform sampling techniques are studied and analyzed by comparison of the uniformity of distributions.

Keywords: Sampling, Spherical Sampling Pattern, Uniform Separation Sampling, Generalized Discrepancy, Monte Carlo Image Rendering.

PACS: 06.60.Ei Sample preparation; 06.60.Mr Testing and inspecting procedures; 06.60.Sx Positioning and alignment; manipulating.
MSC2010: 11K38 Irregularities of distribution, discrepancy; 11P21 Lattice points in specified regions; 11N64 Other results on the distribution of values or the characterization of arithmetic functions; 35B36 Pattern formation; 68U05 Computer graphics; 65C05 Monte Carlo methods.

INTRODUCTION

Sampling of hemisphere and sphere is a fundamental task at Monte Carlo solution of image rendering problems, where the rendering equation [1] is solved numerically. The rendering equation mathematically describes the light propagation in a scene. The radiance L, leaving from a point x on the surface of the scene in direction $\omega \in \Omega_x$, where Ω_x is the hemisphere at point x, is the sum of the self radiating light source radiance L^e and all reflected radiance: $L(x,\omega) = L^e(x,\omega) + \int_{\Omega_x} L(-\omega') f_r(-\omega', x, \omega) \cos\theta' d\omega'$. The radiance L^e has non-zero value if the considered point x is a point from solid light source. Therefore, the reflected radiance in direction ω is an integral of the radiance incoming from all points, which can be seen through the hemisphere Ω_x at point x attenuated by the surface BRDF (Bidirectional Reflectance Distribution Function) $f_r(-\omega', x, \omega)$ and the projection $\cos\theta'$.

When the point x is on a transparent object the transmitted light component must be added to the rendering equation. This component estimates the total light transmitted trough the object and incoming to the point x from all directions opposite to the hemisphere Ω_x. The transmitted light in direction ω is an integral similar to the the reflected radiance integral where the domain of integration is the hemisphere $\overline{\Omega}_x$ at point x and BRDF is substituted by the surface BTDF (Bidirectional Transmittance Distribution Function) [2]. In this case the integration domain for solving the rendering equation is a sphere $\Omega^{(x)}$ at point x, where $\Omega^{(x)} = \Omega_x \bigcup \overline{\Omega}_x$.

Let us consider a few classical sampling techniques [3] for Monte Carlo solution of the rendering equation. Each one uses random variables $u, v \in [0,1]$ to generate different sampling patterns for integration domain. **Cosine weighted** random sampling uses $\varphi = 2\pi u_i, \quad \theta = \arcsin\sqrt{v_i}, \quad u_i, v_i \in [0,1], \quad i = 1, \ldots, N$ to generate a sampling pattern.

Systematic and **Stratified sampling** methods are very similar at all. Let n_1 and n_2 are arbitrary integers, where $n_1 \times n_2 = N$ and $t_1 = \frac{2\pi}{n_1}$ and $t_2 = \frac{1}{n_2}$. The only difference is that **Systematic** generates once $u, v \in [0,1]$, whereas **Stratified** generate pairs (u_i, v_j), where $u_i, v_j \in [0,1]$, $i = 0, \ldots, n_1$, $j = 0, \ldots, n_2$. Then, both methods do the following:

for $i = 0$ **to** $(n_1 - 1)$	**for** $i = 0$ **to** $(n_1 - 1)$
$\varphi_i = t_1(u + i)$	$\varphi_i = t_1(u_i + i)$
for $j = 0$ **to** $(n_2 - 1)$	**for** $j = 0$ **to** $(n_2 - 1)$
$\theta_j = \arcsin\sqrt{t_2(v + j)}$	$\theta_j = \arcsin\sqrt{t_2(v_j + j)}$

A class of ***Uniform Separation*** sampling methods for Monte Carlo solving the rendering equation is introduced by us in [4], further developed and extended in [5] and [6]. **Uniform Triangle Separation**, **Uniform Quadran-**

gle Separation** and **Combined Uniform Separation** ***symmetrically*** partition the spherical integration domain, as shown in Fig. 1, into 48 equal spherical triangles $\Omega_{\triangle}$, 24 equal spherical quadrangles $\Omega_{\square}$ and the combination of 16 equal spherical triangles $\Omega_{\triangle}$ and 16 equal spherical quadrangles $\Omega_{\square}$, respectively. All equal sub-domains are non-overlapped, symmetric each to other and have fixed vertices and computable parameters. We have find the transformations $\left(\varphi_0 = \frac{u\pi}{4}; \theta'_{\triangle_0} = \arctan\frac{v}{\cos\frac{u\pi}{4}}\right)$ and $\left(\varphi_0 = \frac{u\pi}{4}; \theta'_{\square_0} = \mathrm{arccot}\frac{v}{\cos\frac{u\pi}{4}}\right)$, where $u, v \in [0,1]$; $\varphi_0 \in [0, \frac{\pi}{4}]$ and $\theta'_{\triangle_0} \in [0, \arctan\frac{1}{\cos\varphi_0}]$ and $\theta'_{\square_0} \in [\arctan\frac{1}{\cos\varphi_0}, \frac{\pi}{2}]$ for sampling one $\Omega_{\triangle_0}$ and $\Omega_{\square_0}$. The symmetric property allows us to sample only one sub-domain and calculate in parallel the coordinates of the symmetric points on sphere.

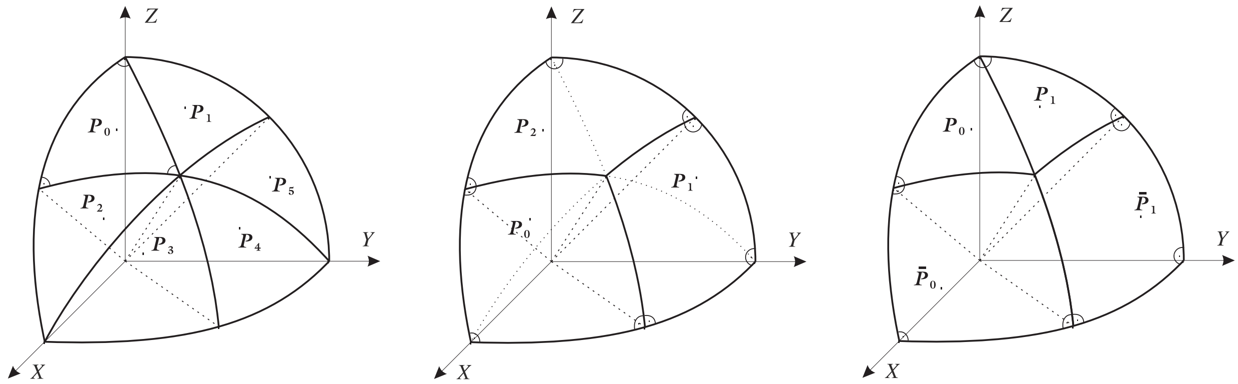

FIGURE 1. **(a)** Uniform Triangle Separation **(b)** Uniform Quadrangle Separation **(c)** Combined Uniform Separation

There is no universal sampling scheme or technique proper for arbitrary scene rendering due to the nature of the rendering equation. Many sampling strategies are design for various tasks and graphical applications. Depending on smoothness of the integrand in rendering equation to be solved numerically, equidistribution of point sets on hemisphere or sphere could be an important advantage leading to reduction of the integration error. Otherwise, equidistribution could be a crucial drawback at discontinuous integrands, but almost all of sampling techniques try to ensure and preserve low discrepancy property of sampling patterns as a possible advantage for Monte Carlo integration.

We compute the discrepancy to analyze and study the equidistribution of point sets on sphere. Examining the above sampling methods, the point sets are generated on hemisphere, then distributed on sphere by inverting the Z coordinate.

GENERALIZED DISCREPANCY FOR POINT SETS ON SPHERE

Peter Shirley in 1991 first introduces the discrepancy as measure into computer graphics. Earlier discrepancy measurements are mainly designed for planar structures. Today, the realistic image creators [7] apply generalized discrepancy to analyze the sampling techniques used in rendering. Generalized discrepancy [8] gives a measure for the uniformity of a point set on sphere.

Definition *Let $\mathbb{A}$ be a pseudo differential operator of order $s, s > 1$, with symbol $\mathbb{A}_n \neq 0$ for $n \geq 1$. Then the generalized discrepancy associated to the operator $\mathbb{A}$ is defined by* $D(\{\eta_1, \ldots, \eta_N\}; \mathbb{A}) = \frac{1}{N}\left[\sum_{j=0}^{j=N}\sum_{i=0}^{i=N}\sum_{n=0}^{n=\infty}\frac{2n+1}{4\pi\mathbb{A}_n^2}P(\eta_i \cdot \eta_j)\right]^{\frac{1}{2}}$.

The generalized discrepancy characterizes ***"how well the point set*** $\{\eta_1, \ldots, \eta_N\}$ ***is equidistributed"***, quote to [7] and [8]. Following the mathematical considerations in [8], we can rewrite the generalized discrepancy as

$$D(N) = \frac{1}{2N\sqrt{\pi}}\left[\sum_{i,j=1}^{N}\left(1 - 2ln\left(1 + \sqrt{\frac{1 - \overrightarrow{P_i} \cdot \overrightarrow{P_j}}{2}}\right)\right)\right]^{\frac{1}{2}},$$

where $\{\overrightarrow{P_1}, \ldots, \overrightarrow{P_N}\}$ is a N-point sequence and each $\overrightarrow{P_i}$ is a point on sphere. Note, in our case of unit sphere, each $\overrightarrow{P_i}(X_i, Y_i, Z_i)$ is unit vector, therefore we obtain $\overrightarrow{P_i} \cdot \overrightarrow{P_j} = |\overrightarrow{P_i}||\overrightarrow{P_j}|\cos(\overrightarrow{P_i} \cdot \overrightarrow{P_j}) = \cos(\overrightarrow{P_i} \cdot \overrightarrow{P_j}) = X_iX_j + Y_iY_j + Z_iZ_j$, as well as $X_i = \cos\varphi_i \sin\theta_i$, $Y_i = \sin\varphi_i \sin\theta_i$ and $Z_i = \cos\theta_i$. The lower the $D(N)$ is, the more uniformly distributed the sampling pattern is, in general $\lim_{N\to\infty} D(N) = 0$.

EXPERIMENTAL RESULTS

We use four different random number generators (RNG) for generation of random points in unit square: RNG-1 is Lagged Fibonacci Generator; RNG-2 is Mersenne Twister Generator; RNG-3 is Standard Random Generator (drand48); and Halton is Halton low discrepancy sequence (LDS) at base 2 and base 3 for the one and other dimension respectively. Different spherical sampling patterns are generated for each of examined methods trough transformation of the uniformly distributed random points from unit square onto sphere point sets . The generalized discrepancy is calculated in each one case for point sets, starting from 96 points and increasing to 12288 points. Numerical results for the generalized discrepancy $D(N)$, computed at different sampling methods are shown in Fig. 2.

Cosine Weighted Sampling				
Number of points, N	RNG-1	RNG-2	RNG-3	Halton
96	0,142024	0,137921	0,132274	0,141084
192	0,138858	0,128592	0,136229	0,139375
384	0,138686	0,132309	0,137505	0,138450
768	0,136305	0,136905	0,135237	0,136853
1536	0,133870	0,135971	0,134189	0,136614
3072	0,134188	0,134955	0,135666	0,136476
6144	0,135340	0,134908	0,135746	0,136218
12288	0,135366	0,135502	0,135642	0,136147

Systematic Sampling				
Number of points, N	RNG-1	RNG-2	RNG-3	Halton
96	0,262963	0,263058	0,263308	0,263317
192	0,267077	0,267142	0,267321	0,267328
384	0,270363	0,270405	0,270523	0,270527
768	0,272650	0,272675	0,272748	0,272751
1536	0,274472	0,274487	0,274531	0,274533
3072	0,275796	0,275805	0,275831	0,275832
6144	0,276904	0,276909	0,276925	0,276925
12288	0,277775	0,277778	0,277787	0,277788

Stratified Sampling				
Number of points, N	RNG-1	RNG-2	RNG-3	Halton
96	0,263746	0,263545	0,263712	0,263268
192	0,267965	0,267735	0,267370	0,267638
384	0,270870	0,270808	0,270755	0,270813
768	0,272917	0,272897	0,273018	0,272972
1536	0,274673	0,274660	0,274663	0,274669
3072	0,275926	0,275926	0,275917	0,275923
6144	0,276984	0,276988	0,276980	0,276979
12288	0,277825	0,277824	0,277825	0,277823

Uniform Triangle Separation Sampling				
Number of points, N	RNG-1	RNG-2	RNG-3	Halton
96	0,036180	0,021756	0,018914	0,026737
192	0,028182	0,013629	0,008618	0,011251
384	0,023406	0,011911	0,007320	0,014694
768	0,016064	0,011628	0,006297	0,014199
1536	0,012926	0,009030	0,006958	0,011980
3072	0,013101	0,007243	0,011391	0,010144
6144	0,012474	0,004608	0,009347	0,009410
12288	0,011169	0,008394	0,009550	0,009389

Uniform Quadrangle Separation Sampling				
Number of points, N	RNG-1	RNG-2	RNG-3	Halton
96	0,015753	0,022718	0,021578	0,015325
192	0,010043	0,011615	0,019524	0,007437
384	0,009084	0,013299	0,017059	0,006745
768	0,010309	0,012414	0,014658	0,006695
1536	0,007744	0,012279	0,008967	0,008111
3072	0,007467	0,012833	0,010345	0,008486
6144	0,008070	0,009256	0,008996	0,008418
12288	0,009872	0,008885	0,010009	0,008584

Combined Uniform Separation Sampling				
Number of points, N	RNG-1	RNG-2	RNG-3	Halton
96	0,032563	0,027892	0,027532	0,045564
192	0,038402	0,033023	0,034795	0,044023
384	0,035110	0,023014	0,035987	0,040338
768	0,030809	0,025412	0,031662	0,034425
1536	0,034570	0,029741	0,033437	0,034704
3072	0,032780	0,029052	0,030560	0,034153
6144	0,034689	0,032040	0,030311	0,033755
12288	0,033320	0,033706	0,030831	0,033299

FIGURE 2. Numerical results for $D(N)$

CONCLUSION

The curves in Fig. 3 and Fig. 4 plot the experimental results of generalized discrepancy with respect to the different RNG. **Series 1-6** correspond to each examined sampling method: **Cosine weighted**, **Systematic** and **Stratified**, **Uniform Triangle Separation**, **Uniform Quadrangle Separation** and **Combined Uniform Separation**, respectively. One can see the values of $D(N)$ are relatively low at all sampling methods, even at $N = 96$. **Uniform Triangle**, **Uniform Quadrangle** and **Combined Uniform Separation** methods explore the uniform sphere partition to strength and achieve more equidistribution of point sets. The best of three results, we obtain at **Uniform Quadrangle Separation** with Halton LDS due to the nature of **Uniform Separation** strategy. This fact directs us to use these sampling methods at Monte Carlo image rendering, where the uniformity of sampling points is an essential advantage. The efficiency of Monte Carlo image rendering is sensitive to the used sampling scheme, as well as to applied RNG. The sensitivity analysis to find efficient sampling for Monte Carlo image rendering is subject of our permanent future work and study.

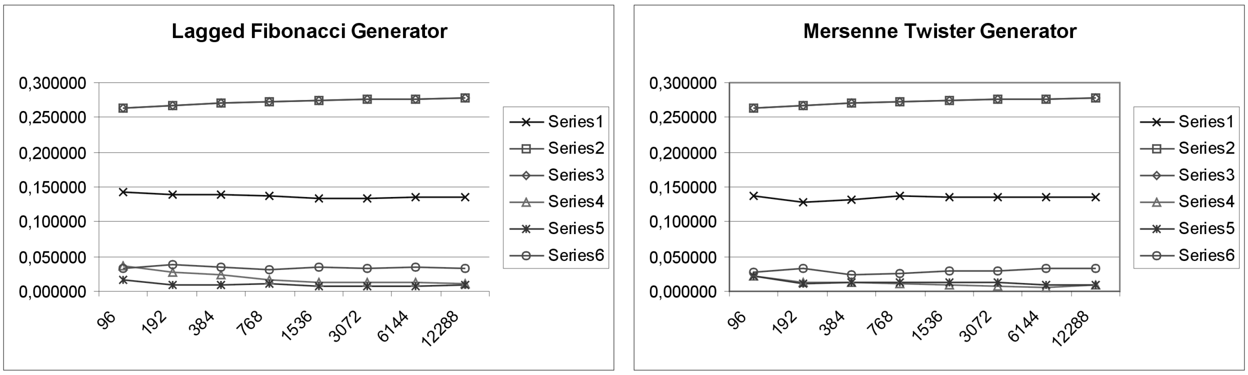

FIGURE 3. Comparative results for **(a) Lagged Fibonacci Generator** and **(b) Mersenne Twister Generator**

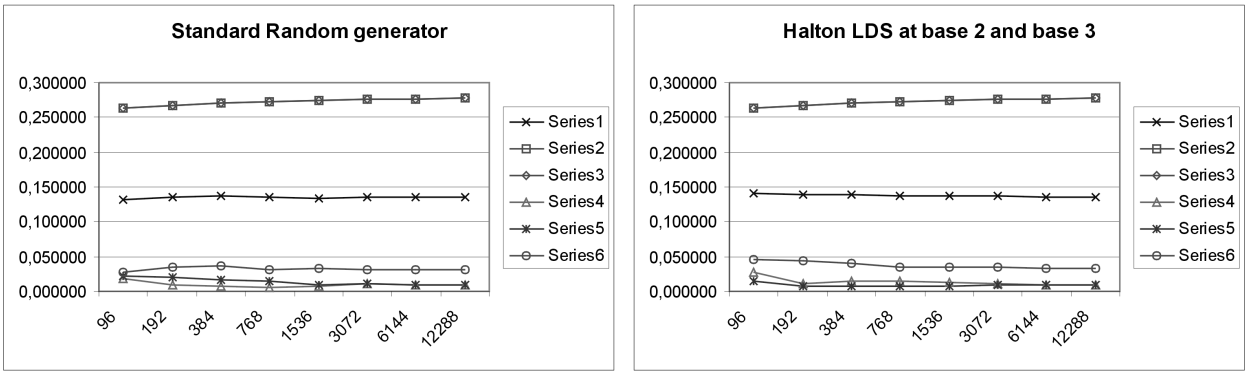

FIGURE 4. Comparative results for **(a) Standard Random Generator (drand48)** and **(b) Halton LDS at base 2 and base 3**

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